THE METHOD OF APPROXIMATIONS AND SKEW PRODUCT TRANSFORMATIONS

by

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CONTENTS

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DECLARATION

The following parts of this thesis will appear in the Journal of the London Mathematical Society under the title "On Spectral Properties of Skew Products over Irrational Rotations":

(a) Theorem II. 2.5 restricted to probability spaces;

(b) the results of Section II. 4;

(c) the results of Section III. 2.

The contents of Sections II. 3 and III. 4 will be submitted for publication as a joint paper-with Giles Atkinson as co-author.

SUMMARY

This thesis presents some extensions and applications of the method of approximations of ergodic theory (see [11,12]).

In the first chapter we define two notions of approximation which are applicable to any σ-finite-measure-preserving group action. We then extend those results of [2], [23] and [12] which relate the speed of such approximations to questions of spectral multiplicity, spectral type and ergodicity. For our result on spectral multiplicity, we first establish a general result on the spectral decomposition of unitary representations (see I. 1).

In Chapter II, we consider applications of the method of approximations to a class of unitary operators arising naturally in the spectral analysis of skew product transformations. These operators, $V_{T,\rho}$, are defined as follows: if T is any measure-preserving automorphism of a σ-finite measure space $(X,\mathcal{B},\mu)$, and $\rho : X \to \{z \in \mathbb{C} : |z| = 1\}$ is a measurable function, then

$$V_{T,\rho}\, y(x) = \rho(x).y(Tx) \quad , \text{ for } y \in L^2(X,\mu), \; x \in X.$$

Relying heavily on a result of [7], we prove a result on the spectral multiplicity of operators of the form $V_{T,\rho}$. A result concerning the singularity of spectrum is also presented.

Chapter II also contains an elementary result concerning the computation of essential values (see [22]), and an investigation of the dependence on ρ of the discrete part of the spectrum of $V_{T,\rho}$.

The last chapter is concerned with applications to skew product transformations based on irrational rotations of the circle. Section III. 2 deals with a class of Z_2-extensions of these rotations, while III. 3 and III. 4 are concerned with classes of cylinder transformations which arise in connection with irregularity of distribution (see [21]). These cylinder transformations provide natural examples of approximations of all finite multiplicities (on infinite measure spaces). The method of approximations is shown to be a natural tool for the study of their spectral and ergodic properties.

INTRODUCTION

In their fundamental papers [11,12], Katok and Stepin introduced the "method of approximations" into ergodic theory. In so doing they showed that many properties of a measure-preserving automorphism of a probability space may be deduced from the existence of certain types of approximation by periodic transformations.

This thesis presents some extensions and applications of those aspects of the method of approximations which deal with the determination of spectral properties.

The notions of approximation which we shall consider are introduced in Section I. 2. One, that of a finite multiplicity approximation, is a development of an idea of Chacon [2]. The other, the notion of a cyclic approximation, derives directly from [12]. Both of these types of approximation are defined for the first time in such a way that they may be applied to any Borel group action which preserves a σ-finite measure.

Chapter I is devoted to a study of how, in this general setting, the method of approximations is useful in the investigation of spectral properties. The main results of this study are presented in detail in Section I. 3. They may be summarized as follows:

(a) Theorem I. 3.1 relates the spectral multiplicity of an action of a type I group to the speed with which it admits finite multiplicity approximations;

(b) Theorem I. 3.4 shows that if an action of an abelian group admits a certain speed of cyclic approximation, then it has singular spectral type;

(c) Theorem I. 3.6 relates the ergodicity of an action of a general group to the speed which it admits cyclic approximations.

These theorems are generalizations of results of Chacon [2], Stepin [23] and Katok and Stepin [12], respectively.

In order to establish I. 3.1, it is first necessary to prove a result in the spectral decomposition theory of unitary group representations. This is done in I.1 (see Proposition I. 1.1), a preliminary section which includes a brief introduction to the terminology of spectral multiplicity theory.

In Chapter II, we specialize, and consider the application of the method of approximations to skew product transformations. The preliminary Section, II. 1, shows, after a few definitions, that the spectral analysis of such transformations is closely related to that of the unitary operators $V_{T,\rho}$ defined as follows (c.f. Definition II. 1.4): if T is any measure-preserving automorphism of a σ-finite measure space $(X,\mathcal{B},\mu)$, and $\rho : X \to \{z \in \mathbb{C} : |z| = 1\}$ is a measurable function, then

$$V_{T,\rho}\, y(x) = \rho(x).y(Tx), \text{ for all } y \in L^2(X,\mu) \text{ and } x \in X.$$

Section II. 2 is devoted to establishing counterparts of Theorems I. 3.1 and I. 3.4 for operators of the form $V_{T,\rho}$. One of the resulting theorems, II. 2.2, builds upon a result of Goodson [7].

In investigating the ergodicity of skew product transformations, the concept of essential value is fundamental (see Schmidt [22]). In II. 3, after defining what is meant by the essential values of a function, we present an elementary proposition providing a method for their computation.

Section II. 4 is an exposition of some simple observations concerning how, for fixed T, the discrete part of the spectrum of $V_{T,\rho}$ may vary with the choice of the function ρ.

The last chapter consists of applications of the results of Chapters I and II. All of the examples considered are skew products based on irrational rotations of the circle. Section III. 2 deals with a class of Z_2-extensions of these rotations, while Sections III. 3 and III. 4 are concerned with certain classes of cylinder transformations ($\mathbb{R}$ and Z-extensions of irrational rotations). These examples, which are also of interest in connection with the irregularity of distribution of the sequences $n\alpha$, $n = 1,2,\ldots$, α irrational (see Veech [24]or Schmidt [21], for example), provide natural examples of approximations of all finite multiplicities. As a particularly interesting example of the results of Chapter III, let us mention that certain of the cylinder transformations considered are shown to be ergodic, with singular spectral type and spectral multiplicity uniformly equal to two (see III. 3.8, and apply III. 3.4 and III. 4.4 to the transformations $S_{\alpha,\frac{1}{2}}$).

Finally, as an illustration of our numbering scheme, note the following: Theorem I. 3.4 is the fourth of the numbered elements (theorems, definitions, remarks, etc.) in Section I.3. Within Chapter I, this theorem would be called Theorem 3.4, while Section I. 3 would be referred to merely as Section 3.

CHAPTER I

APPROXIMATIONS AND THE SPECTRAL PROPERTIES OF MEASURE-PRESERVING GROUP ACTIONS - GENERAL RESULTS

§1. Preliminaries - A Result in Spectral Theory

Let X be a standard Borel space equipped with a σ-finite measure μ. Suppose that some locally compact, second countable group G is known to act in a certain way on X. If this action $(g,x) \to g.x$, $g \in G$, $x \in X$, is Borel and measure-preserving, then it induces a unitary representation Π_g, $g \in G$, defined on the separable Hilbert space $L^2(X,\mu)$ as follows:

$$(\Pi_g y)(x) = y(g^{-1}.x), \text{ for all } y \in L^2(X,\mu),\ x \in X.$$

If G is assumed to be a type I group, then every such induced representation (in fact, every continuous unitary representation of G on a separable Hilbert space) admits a canonical spectral decomposition

$$\Pi \cong \int_{\hat{G}}^{\oplus} m(\lambda).\lambda \quad d\nu(\lambda) \qquad (*),$$

where

(a) $\hat{G}$ denotes the standard Borel space of unitary equivalence classes of irreducible representations of G,

(b) m is a measurable function on $\hat{G}$ with values in $\{\infty; 1,2,\ldots\}$, known as the spectral multiplicity function of the measure-preserving G-action (or of the representation Π),

(c) $m(\lambda).\lambda$ denotes the direct sum of $m(\lambda)$ copies of the irreducible λ, for all $\lambda \in \hat{G}$, and

(d) ν is a σ-finite, Borel measure on $\hat{G}$, known as a maximal spectral measure of the measure-preserving G-action (or of the representation Π).

Note that (*) determines the pair (ν, m) uniquely up to equivalence of measures and almost everywhere equality of functions. The equivalence class of the measure ν is called the maximal spectral type of the measure-preserving G-action (or of Π).

For more details concerning the spectral decomposition of measure-preserving group actions, see Kirillov [14]. Mackey [17] and Dixmier [5] deal with the spectral decomposition of general unitary group representations.

At this point, it is convenient to introduce the following notation: given a type I group G, then, for each $k \in \{\infty; 1,2,...\}$, $\hat{G}_k$ denotes the Borel subset of $\hat{G}$ consisting of all elements λ with $\dim \lambda = k$; given a representation Π defined on a space H, we shall denote by $Z(z)$ the closed, Π-cyclic subspace generated by a vector z in H.

The following proposition, introduced here in readiness for its use in §3, was proved by Chacon [2] in the case of a single unitary operator (i.e. for G = Z). The main innovation in our extension of Chacon's proof to cover the case of a general type I group is Lemma 1.3, below.

PROPOSITION 1.1.

Let Π be a continuous, unitary representation of a type I group G on a separable Hilbert space H. Let m be the spectral multiplicity function for this representation.

Then, with ν denoting a maximal spectral measure of Π,

(a) Π is cyclic if and only if

$$m(\lambda) \leq \dim \lambda, \text{ for } \nu\text{-a.e. } \lambda \text{ in } \hat{G},$$

and

(b) if Π is not cyclic, then, to each pair of positive integers $\ell > k$ with the property that $m(\lambda)$ is greater than or equal to ℓ on a ν-non-null subset of $\hat{G}_k$, there corresponds a choice of ℓ orthonormal vectors $y_1,\ldots,y_\ell$ in H with

$$\sum_{j=1}^{\ell} d^2(y_j, Z(z)) \geq \ell-k, \text{ for all } z \text{ in } H,$$

where d denotes the distance induced from the norm on H.

For the proof of this proposition, we shall need a more specific version of the spectral decomposition (*). Assign to each $k \in \{\infty;1,2,\ldots\}$ a fixed separable Hilbert space H_k of dimension k. The direct sum $m(\lambda).\lambda$ occurring in (*) is unitarily equivalent to the tensor product representation $g \to \lambda_g \otimes Id_{H_{m(\lambda)}}$, $g \in G$, taken to act on the space $H_{\dim\lambda} \otimes H_{m(\lambda)}$. To see this, choose an orthonormal basis $w_1,\ldots,w_{m(\lambda)}$ for $H_{m(\lambda)}$, split the space $H_{\dim\lambda} \otimes H_{m(\lambda)}$ into the orthogonal direct sum of the closed $\lambda \otimes Id_{H_{m(\lambda)}}$ - invariant subspaces $\{v \otimes w_j : v \in H_{\dim \lambda}\}, j=1,\ldots,m(\lambda)$, and observe that the restriction of $\lambda \otimes \mathrm{Id}_{H_{m(\lambda)}}$ to any of these subspaces is isomorphic to λ. Thus, (*) is equivalent to saying that every unitary representation of G is isomorphic to one of the form

$$\Pi^{(\nu,m)} = \int_{\hat{G}}^{\oplus} (\lambda \otimes \mathrm{Id}_{H_{m(\lambda)}})\, d\nu(\lambda),$$

taken to act on the direct integral Hilbert space

$$H^{(\nu,m)} = \int_{\hat{G}}^{\oplus} (H_{\dim\lambda} \otimes H_{m(\lambda)})\, d\nu(\lambda).$$

Now, suppose that $\lambda \to z(\lambda)$, $\lambda \in \hat{G}$, is one of the norm-square-integrable vector fields making up $H^{(\nu,m)}$. Let $P_{Z(z)}$ denote the projection from $H^{(\nu,m)}$ onto the $\Pi^{(\nu,m)}$ - cyclic subspace generated by z. Then, by the $\Pi^{(\nu,m)}$-invariance of $Z(z)$,

$$P_{Z(z)}\, \Pi_g^{(\nu,m)} = \Pi_g^{(\nu,m)}\, P_{Z(z)}, \text{ for all } g \in G.$$

By Dixmier [5], 8.4.1 (noting that in the terminology used by Dixmier, (*) is a version of the "désintégration centrale" of Π), and Theorem 4 of [4], II,3, this commutativity property is sufficient for $P_{Z(z)}$ to be decomposable in the sense of Defintion 2 of [4], II,2. Hence there exists a "measurable" assignment $\lambda \to P_{Z(z)}(\lambda)$, $\lambda \in \hat{G}$, such that

(i) $P_{Z(z)}(\lambda)$ is a projection operator on $H^{(\nu,m)}(\lambda) = H_{\dim\lambda} \otimes H_{m(\lambda)}$, for all $\lambda \in \hat{G}$, and

(ii) for all $y \in H^{(\nu,m)}$, $(P_{Z(z)} \cdot y)(\lambda) = P_{Z(z)}(\lambda) \cdot y(\lambda)$, ν-a.e. on $\hat{G}$.

Thus, we are lead to the following elementary lemma, which will allow us to prove Proposition 1.1 "fibrewise". The proof of this lemma, which uses very routine techniques, is included only because of its seeming non-appearance in the standard references.

LEMMA 1.2.

For all z in $H^{(\nu,m)}$, the projection $P_{Z(z)}$ decomposes as follows:

$$P_{Z(z)}(\lambda) = P_{Z(z(\lambda))}, \quad \nu \text{ - a.e. on } \hat{G},$$

where, for each $\lambda \in \hat{G}$, $P_{Z(z(\lambda))}$ denotes the orthogonal projection from $H_{\dim\lambda} \otimes H_{m(\lambda)}$ onto the closed $\lambda \otimes Id_{H_{m(\lambda)}}$ - cyclic subspace generated by $z(\lambda)$. (In the notation of [4], this lemma says that $P_{Z(z)} = \int_{\hat{G}}^{\oplus} P_{Z(z(\lambda))}\, d\nu(\lambda)$.)

Proof

The decomposability of $P_{Z(z)}$ implies, using Proposition 9 of [4],II,1, that Z(z) contains a countable set of vector fields $\{y_i : i \in Z\}$ with the property that $\{y_i(\lambda) : i \in Z\}$ is dense in the range of $P_{Z(z)}(\lambda)$, for ν-almost every λ in $\hat{G}$. Corresponding to each $y_i \in Z(z)$ there is a sequence $T_1^{(i)}, T_2^{(i)}, \ldots$ of linear combinations of the elements of $\{\Pi_g^{(\nu,m)} : g \in G\}$ such that $y_i = \lim_{n\to\infty} T_n^{(i)} z$. Using Proposition 5 of [4], II, 1, and replacing $T_1^{(i)}, T_2^{(i)}, \ldots$ by a subsequence, it may be assumed, for each $i \in Z$, that $y_i(\lambda) = \lim_{n\to\infty} T_n^{(i)}(\lambda).z(\lambda)$, ν-a.e. on $\hat{G}$. Since there are only countably many y_i's, and each $T_n^{(i)}(\lambda)$ is a linear combination of the elements of $\{\lambda_g \otimes Id_{H_{m(\lambda)}} : g \in G\}$, it follows that, for ν-a.e. λ in $\hat{G}$, all of the vectors $y_i(\lambda)$, $i \in Z$, belong to $Z(z(\lambda))$. The defining property of $\{y_i : i \in Z\}$ now implies that

$$P_{Z(z)}(\lambda) \leq P_{Z(z(\lambda))}, \quad \nu\text{-a.e on } \hat{G}. \tag{1}$$

On the other hand, since $P_{Z(z)} \Pi_g^{(\nu,m)} z = \Pi_g^{(\nu,m)} z$, for all $g \in G$, it follows routinely from the separability of G that, for ν -a.e. λ in $\hat{G}$,

$$P_{Z(z)}(\lambda)(\lambda_g \otimes Id_{H_{m(\lambda)}}). z(\lambda)$$

$$= (\lambda_g \otimes Id_{H_{m(\lambda)}}).z(\lambda), \text{ for all } g \in G.$$

This implies that

$$P_{Z(z(\lambda))} \leq P_{Z(z)}(\lambda), \quad \nu\text{-a.e. on } \hat{G}. \tag{2}$$

Combining (1) and (2), one obtains the conclusion of the lemma.

The following lemma proves Proposition 1.1 in the case when the maximal spectral type of Π is concentrated at a single point.

<u>LEMMA 3.1.</u>

Let Π be the tensor product representation $\lambda \otimes Id_{H_\ell}$ defined on the space $H_k \otimes H_\ell$, where $\lambda \in \hat{G}_k$, and k and ℓ belong to $\{\infty;1,2,...\}$.

Then

(a) Π is cyclic if and only if $\ell \leq k$ (in particular, whenever $k = \infty$),

and

(b) if Π is not cyclic (i.e. if $\ell > k$), then, given any orthonormal basis $w_1,\ldots,w_\ell$ for H_ℓ (respectively a complete orthonormal sequence $w_1, w_2,\ldots$ in H_ℓ, if $\ell = \infty$), and any choice of ℓ unit vectors $v_1,\ldots,v_\ell$ in H_k (respectively a sequence of unit vectors $v_1,v_2,\ldots$ in H_k, if $\ell = \infty$), the inequality

$$\sum_{j=1}^{\ell} d^2(v_j \otimes w_j, Z(z)) \geq \ell-k \ (=\infty, \text{ if } \ell = \infty)$$

is satisfied by all z in $H_k \otimes H_\ell$.

<u>Proof</u>

Choose an orthonormal basis $u_1,\ldots,u_k$ for H_k (respectively a complete orthonormal sequence $u_1,u_2,\ldots$, if $k = \infty$). The tensor product space $H_k \otimes H_\ell$ splits into the orthogonal direct sum of the closed subspaces $\{u_i \otimes w : w \in H_\ell\}$, $i = 1,\ldots,k$ (respectively $i = 1,2,\ldots$, if $k = \infty$). Hence, each vector z in $H_k \otimes H_\ell$ has a unique decomposition as a sum

$$z = \sum_{i=1}^{k} u_i \otimes z_i, \text{ with } z_i \in H_\ell, \text{ for } i = 1,\ldots,k$$

(if $k = \infty$, then each z corresponds to a sequence of vectors $z_1,z_2,\ldots$ in H_ℓ with $\sum_{i=1}^{\infty} \|z_i\|^2 < \infty$). So, for each z in $H_k \otimes H_\ell$, one may write

$$\Pi_g z = \sum_{i=1}^{k} (\lambda_g u_i) \otimes z_i, \text{ for all } g \in G.$$

From this expression and the irreducibility of $\lambda \in \hat{G}_k$, it follows that, for arbitrary z in $H_k \otimes H_\ell$,

$$(1)\ldots \quad Z(z) = \begin{cases} H_k \otimes \text{lin.span}\,\{z_1, z_2, \ldots, z_k\}, & \text{if } k < \infty, \\ H_k \otimes \overline{\text{lin.span}\,\{z_1, z_2, \ldots\}}, & \text{if } k = \infty. \end{cases}$$

Thus, the k-tuples (respectively sequences, if $k = \infty$) of vectors which span dense subspaces of H_ℓ correspond to the cyclic vectors for Π. This proves (a), since the condition $\ell \leq k$ is necessary and sufficient for the existence of such k-tuples (respectively sequences).

For general z in $H_k \otimes H_\ell$, (1) implies that the orthogonal projection with range Z(z) is just $\mathrm{Id}_{H_k} \otimes P$, where P denotes the projection from H_ℓ onto the closed subspace generated by $z_1, \ldots, z_k$ (respectively $z_1, z_2, \ldots$, if $k = \infty$). Hence, given any orthonormal basis $w_1, \ldots, w_\ell$ for H_ℓ, and unit vectors $v_1, \ldots, v_\ell$ in H_k, one has

$$\sum_{j=1}^{\ell} d^2(v_j \otimes w_j,\ Z(z))$$

$$= \sum_{j=1}^{\ell} \|v_j \otimes w_j - v_j \otimes Pw_j\|^2$$

$$= \sum_{j=1}^{\ell} \|v_j\|^2\ \|(\mathrm{Id}_{H_\ell} - P)w_j\|^2$$

$$= \text{trace}\,(\mathrm{Id}_{H_\ell} - P).$$

Thus, the value of the quantity $\sum_{j=1}^{\ell} d^2(v_j \otimes w_j, Z(z))$ at a point $z = \sum_{i=1}^{\ell} u_i \otimes z_i$ in $H_k \otimes H_\ell$ is just the codimension in H_ℓ of the closed subspace generated by $z_1,\dots,z_k$ (respectively $z_1, z_2, \dots$, if $k = \infty$).

For $\ell > k$ (note that this excludes the case $k = \infty$), this codimension is bounded away from zero by $\ell - k$. This completes the proof of the lemma.

The Proof of Proposition 1.1

(a) Since Π is unitarily equivalent to the representation $\Pi^{(\nu,m)}$, part (a) of the proposition is an immediate consequence of the following two results, taken from Lemma 1.2 and Lemma 1.3(a), respectively: $\Pi^{(\nu,m)}$ is cyclic if and only if ν-almost all of the "fibre" representations $\Pi^{(\nu,m)}(\lambda) = \lambda \otimes \mathrm{Id}_{H_{m(\lambda)}}$, $\lambda \in \hat{G}$, are cyclic; for any $\lambda \in \hat{G}$, the representation $\lambda \otimes \mathrm{Id}_{H_{m(\lambda)}}$ is cyclic if and only if $m(\lambda) \le \dim \lambda$.

(b) From (a) it follows that if Π is not cyclic, then there must exist a pair of positive integers, k and ℓ, with $\ell > k$ and $\nu\{\lambda \in \hat{G}_k : m(\lambda) \ge \ell\} > 0$. Fix such k and ℓ, and let

$$B_o = \{\lambda \in \hat{G}_k : m(\lambda) \ge \ell\}.$$

Normalize the maximal spectral measure so that $\nu(B_o) = 1$, and let ν' denote the restriction of ν to B_o, defined as follows:

$\nu'(B) = \nu(B \cap B_o)$, for all Borel $B \subset \hat{G}$. Let m' denote the multiplicity function defined by letting

$$m'(\lambda) = \ell, \qquad \text{for all } \lambda \in \hat{G}.$$

Then, since ν' is absolutely continuous with respect to ν, and $m'(\lambda)$ is less than or equal to $m(\lambda)$, ν'-a.e. on $\hat{G}$, the representation

$$\Pi^{(\nu',m')} = \int_G^{\oplus} \lambda \otimes \mathrm{Id}_{H_{m'(\lambda)}} \, d\nu'(\lambda)$$

is isomorphic to the restriction of Π to some invariant subspace of H. Let Q denote the orthogonal projection onto such a subspace H'. Then, for $y \in H'$, $z \in H$ and T a linear combination of the elements of $\{\Pi_g : g \in G\}$, one has

$$\begin{aligned} \|y - Tz\|^2 &\geq \|Q(y - Tz)\|^2 \\ &= \|y - TQz\|^2 \\ &\geq d^2(y, Z(Qz)). \end{aligned}$$

Taking the infimum over all possible finite linear combinations T, it follows that if $y \in H'$, then $d^2(y,Z(z)) \geq d^2(y, Z(Qz))$, for all z in H.

Thus, it will be sufficient to show that there exist ℓ orthonormal vectors $y_1,\ldots,y_\ell$ in H' with $\sum_{j=1}^{\ell} d^2(y_j,Z(z)) \geq \ell-k$, for all z in H'. To prove this is equivalent to proving (b) in the special case when $H = H^{(\nu',m')}$ and $\Pi = \Pi^{(\nu',m')}$.

Letting $w_1,\ldots,w_\ell$ be an orthonormal basis for H_ℓ, and letting $v_1,\ldots,v_\ell$ be unit vectors in H_k, we choose as our ℓ orthonormal elements of $H^{(\nu',m')}$ the vector fields $\lambda \to y_j(\lambda)$, $\lambda \in \hat{G}$, $j=1,\ldots,\ell$, defined as follows:

$$y_j(\lambda) = \begin{cases} v_j \otimes w_j & , \quad \text{for all } \lambda \in B_o, \\ 0 & , \quad \text{for all } \lambda \in \hat{G}\backslash B_o . \end{cases}$$

Then, for all $z \in H^{(\nu',m')}$,

$$\sum_{j=1}^{\ell} d^2(y_j, Z(z))$$

$$= \sum_{j=1}^{\ell} \| y_j - P_{Z(z)} y_j \|^2$$

$$= \sum_{j=1}^{\ell} \int_{\hat{G}} \| y_j(\lambda) - P_{Z(z(\lambda))} y_j(\lambda) \|^2 d\nu'(\lambda))$$

(by Lemma 1.2 applied to $\Pi^{(\nu',m')}$)

$$= \int_{B_o} \Big(\sum_{j=1}^{\ell} d^2(v_j \otimes w_j, Z(z(\lambda))) \Big) d\nu'(\lambda)$$

(by the definition of $y_1, \ldots, y_\ell$).

Since $B_o \subset \hat{G}_k$ and $\Pi^{(\nu',m')}(\lambda) = \lambda \otimes Id_{H_\ell}$, for all λ in B_o, it follows from Lemma 1.3(b) that the integrand in the above expression is greater than or equal to $\ell-k$ everywhere on B_o. From this and the fact that ν' is a probability measure on B_o, one obtains the required result.

Remark 1.4.

From its proof it is clear that the inequality of Proposition 1.1(b) is precise in the sense that, for the given choice of $y_1, \ldots, y_\ell$, there always exists a vector z in H with $\sum_{j=1}^{\ell} d^2(y_j, Z(z)) = \ell - k$ (assuming, of course, that $\ell > k$).

§2. The Method of Approximations - Basic Definitions

Fix a standard Borel space X, equipped with a σ-finite measure μ.

Definition 2.1. (c.f. del Junco [10])

(a) A semi-partition (of X) is a countable collection of pairwise disjoint, non-null, measurable subsets of X.

(b) A partition (of X) is a semi-partition whose elements cover all of X.

(c) Two semi-partitions ξ and ξ' shall be described as disjoint if$\left(\bigcup_{C\in\xi} C\right)$ and $\left(\bigcup_{C'\in\xi'} C'\right)$ are disjoint subsets of X. In this case, $\xi \vee \xi'$ shall denote the semi-partition which contains all the elements of both ξ and ξ'.

Defintion 2.2.

Let ξ be a semi-partition of X. Then, given any measurable $B \subset X$, $B(\xi)$ denotes the (possibly empty) union of those elements of ξ which are more than half-contained in B (i.e. those $C \in \xi$ for which $\mu(C \cap B) > \mu(C)/2$).

Remark 2.3.

If $\mu(B)$ is finite, then, in the sense that it minimizes the measure of the symmetric difference $B \,\Delta\, B(\xi)$, the set $B(\xi)$ is a best approximation to B among unions of the elements of ξ.

Definition 2.4.

A sequence of semi-partitions $\xi(n)$, $n = 1,2,..$ is said to converge to the unit partition, written $\xi(n) \to \varepsilon$, as $n \to \infty$, if $\lim_{n\to\infty} \mu(B \,\Delta\, B(\xi(n))) = 0$, whenever B is a measurable subset of X of finite measure.

Let there be given a measure (μ) - preserving, Borel action, $(g,x) \to g.x$, $g \in G$, $x \in X$, of a locally compact second countable group G on the space X.

In the definitions that follow, N denotes a positive integer and $f(n)$, $n = 1,2,...$ a sequence of non-negative real numbers.

Definition 2.5.

We shall say that a measure-preserving action of G on X admits a multiplicity N approximation with speed $f(n)$ if there exists a sequence of finite semi-partitions $\xi(n)$, $n = 1,2,...$ such that

(i) $\xi(n) \to \varepsilon$, as $n \to \infty$;

and such that, for each n, there is a splitting

$$\xi(n) = \xi_1(n) \vee \xi_2(n) \vee \ldots \vee \xi_N(n)$$

into N mutually disjoint component semi-partitions

$$\xi_j(n) = \{C_{ij}(n) : i = 1,\ldots,q_j(n)\} , \; j = 1,\ldots,N,$$

for each of which

(ii) all of the elements are of the same finite measure, i.e. $\mu(C_{ij}(n)) = \mu(C_{1j}(n)) < \infty$, for all $i=1,\ldots,q_j(n)$, and

(iii) group elements $g_{1j}(n),\ldots,g_{q_j(n)-1,j}(n)$ may be chosen so that

$$(1/(q_j(n)\mu(C_{1j}(n)))).\sum_{i=1}^{q_j(n)-1} \mu(g_{ij}(n).C_{ij}(n) \,\Delta\, C_{i+1,j}(n))$$

$$\leq f(q_j(n)).$$

The following defines a special type of multiplicity one approximation.

Definition 2.6.

We shall say that a measure-preserving action on G on X admits a cyclic approximation with speed f(n) if there exists a sequence of finite semi-partitions

$$\xi(n) = \{C_1(n),\ldots,C_{q(n)}(n)\},\ n = 1,2,\ldots\ ,$$

such that

(i) $\xi(n) \to \varepsilon$, as $n \to \infty$,

(ii) in each $\xi(n)$, all of the elements are of the same finite measure, i.e. $\mu(C_i(n)) = \mu(C_1(n)) < \infty$, for all $i=1,\ldots,q(n)$, and

(iii) for each $n = 1,2,\ldots$, group elements $g_1(n),g_2(n),\ldots,g_{q(n)}(n)$ may be chosen so that

$$(1/(q(n)\mu(C_1(n)))).\sum_{i=1}^{q(n)} \mu(g_i(n).C_i(n) \,\Delta\, C_{i+1}(n)) \leq f(q(n)),$$

where $C_{q(n)+1}(n)$ is taken to denote $C_1(n)$.

Now, let T be a measure-preserving automorphism of the space X. Using Definitions 2.5 and 2.6, one could seek approximations of the Z-action generated by such an automorphism. However, the following more restrictive definitions will prove to be better suited to applications to skew product transformations (see Chapters II and III).

Definition 2.5'. (c.f. Definition 2.4. of [2]).

A measure-preserving automorphism T is said to admit a multiplicity N approximation with speed f(n) if there exists a sequence of finite semi-partitions

$$\xi(n) = \xi_1(n) \vee \xi_2(n) \vee \dots \vee \xi_N(n)$$

$$= \{C_{ij}(n) : i = 1,\dots,q_j(n);\ j = 1,\dots,N\},\ n = 1,2,\dots$$

satisfying the conditions of Definition 2.5, with the exception of (iii), which is replaced by:

$$\text{(iii)}'\quad (1/(q_j(n)\mu(C_{1j}(n)))) \cdot \sum_{i=1}^{q_j(n)-1} \mu(TC_i(n) \,\Delta\, C_{i+1}(n)) \le f(q_j(n)),$$

for all n = 1,2,..., and j = 1,...,N.

Definition 2.6'. (c.f. Definition 1.1. of [12]).

A measure-preserving automorphism T is said to admit a cyclic approximation with speed f(n) if there exists a sequence of finite semi-partitions

$$\xi(n) = \{C_i(n) : i = 1,2,\dots,q(n)\},\ n = 1,2,\dots$$

satisfying the conditions of Definition 2.6, with the exception that (iii) be replaced by:

$$\text{(iii)}' \quad (1/(q(n)\mu(C_1(n)))).\sum_{i=1}^{q(n)} \mu(TC_i(n) \,\Delta\, C_{i+1}(n))$$

$$\le f(q(n)), \text{ for all } n,$$

where, as before, $C_{q(n)+1}(n)$ is taken to denote $C_1(n)$.

Remark 2.7.

Conditions (i) and (ii) of Definition 2.6 (respectively 2.6') imply that

$$\lim_{n\to\infty} q(n)\mu(C_1(n)) = \mu(X).$$

In the case when μ is finite, the normalizing factor $(1/q(n)\mu(C_1(n))))$ in condition (iii) (respectively (iii)') could be replaced by $(1/\mu(X))$ without affecting subsequent results.

Remark 2.8.

The Z-action generated by an irrational rotation of the circle admits cyclic approximations (Definition 2.6), with arbitrary speed. In terms of Definition 2.6', this need not be true of the irrational rotation itself.

For Z-actions, apart from the dropping of the customary hypothesis that μ be a probability measure, the novel feature of Definitions 2.5 and 2.6 is the free choice of group elements which condition (iii) of each of them allows.

We finish with a simple illustration of some of the concepts introduced in this section.

Example 2.9.

Let γ_1 and γ_2 be rationally independent real numbers. Consider the Z^2-action on the real line defined as follows:

$$(k_1,k_2).\ x = x + k_1\,\gamma_1 + k_2\,\gamma_2\ , \quad \text{for all } (k_1,k_2) \in Z^2,$$
$$\text{and } x \in \mathbb{R}.$$

This action preserves the Lebesgue measure, denoted μ .

The rational independence of γ_1 and γ_2 implies that, if C and C' are arbitrary sub-intervals of $\mathbb{R}$ of the same finite length, then, given any $\varepsilon > 0$, arbitrarily large integers k_1 and k_2 may be found so that $\mu((k_1,k_2).C \,\Delta\, C')$ is less than ε. From this it is clear, for the given Z^2-action, that the sequence of semi-partitions

$$\xi(n) = \{[i/n, i+1/n):\ i = -n^2, -n^2+1, \ldots, n^2\},\ n = 1,2,\ldots$$

satisfies condition (iii) of Definition 2.6 - with arbitrary speed. Condition (ii) of that definition is trivially also satisfied. To check condition (i), i.e. that $\xi(n) \to \varepsilon$, as $n \to \infty$, it is sufficient to show that $\lim_{n\to\infty} \mu(B \,\Delta\, B(\xi(n)))$ is equal to zero wherever B is a finite union of bounded intervals. This is obvious, since

(a) any bounded interval is eventually covered by the intervals making up the approximating semi-partitions, and

(b) the common length of the intervals in $\xi(n)$ tends to zero as $n \to \infty$.

Thus, all the conditions of 2.6 are satisfied, and the given Z^2-action has been shown to admit cyclic approximations with arbitrary speed.

§3. Approximations and Spectral Properties

The proof of the following theorem combines techniques of Chacon [2] and Stepin [23].

THEOREM 3.1.

Suppose that a measure (μ)-preserving Borel action, $(g,x) \to g.x$, $g \in G$, $x \in X$, of a type I group, G, on a standard Borel space X admits a multiplicity N approximation with speed θ/n, $0 \leq \theta < 2$. Then its spectral multiplicity function, m, satisfies the inequality

$$m(\lambda) \leq (2N/(2-\theta)).\ \dim \lambda ,$$

for almost every λ in $\hat{G}$ (with respect to the maximal spectral type of the G-action).

Proof

Fix a positive integer, n, and let

$$\xi(n) = \{C_{ij}(n) : i = 1,\ldots,q_j(n);\ j = 1,\ldots,N\}$$

and $g_{ij}(n)$, $i = 1,\ldots,q_j(n)-1, j=1,\ldots,N$, be as in Definition 2.5.

For each $j \in \{1,\ldots,N\}$, define group elements

$$h_{ij}(n) = \begin{cases} g_{i-1,j}(n).\ g_{i-2,j}(n)\ldots g_{1j}(n), & \text{for } i=2,\ldots,q_j(n), \\ e \ \text{(the group identity)}, & \text{for } i = 1, \end{cases}$$

and consider the subset of $C_{1j}(n)$ defined as follows:

$$A_j(n) = \bigcap_{i=1}^{q_j(n)} h_{ij}(n)^{-1}.\, C_{ij}(n).$$

Clearly, $h_{ij}(n).\, A_j(n) \subset C_{ij}(n)$, for all $i = 1,\dots,q_j(n)$. Furthermore, noting that whenever $x \in C_{1j}(n)\backslash A_j(n)$ there must exist a first $i \in \{2,\dots,q_j(n)\}$ such that $x \notin h_{ij}(n)^{-1}.C_{ij}(n)$, we have, for each $i = 1,\dots,q_j(n)$,

$$\mu(C_{ij}(n)\backslash h_{ij}(n).A_j(n))$$

$$= \mu(C_{1j}(n)) - \mu(A_j(n))$$

$$\leq \sum_{i=1}^{q_j(n)=1} \mu(h_{ij}(n)^{-1}.C_{ij}(n)\backslash h_{i+1,j}(n)^{-1}.C_{i+1,j}(n))$$

$$= \frac{1}{2} \sum_{i=1}^{q_j(n)-1} \mu(g_{ij}(n).C_{ij}(n) \,\Delta\, C_{i+1,j}(n))$$

$$\leq (\tfrac{1}{2}).(\theta/q_j(n)).(q_j(n)\ \mu(C_{1j}(n)))$$

(by Definition 2.5)

$$= (\theta/2).\ \mu(C_{ij}(n)). \tag{1}$$

This inequality will be used to show that, with respect to the unitary representation Π induced from the given group action, one has

$$\limsup_{n\to\infty} \sum_{j=1}^{N} d^2(y, Z(\chi_{A_j(n)}))$$
$$\leq (N-1+(\theta/2))\ \|y\|^2, \text{ for all } y \in L^2(X,\mu), \tag{2}$$

where, in accordance with the notation of §1, for each j $Z(\chi_{A_j(n)})$ denotes the Π-cyclic subspace of $L^2(X,\mu)$ generated by the characteristic function of the set $A_j(n)$, and d denotes the distance induced from the norm on $L^2(X,\mu)$.

To obtain (2), fix y, and, for each n, let

$$y(n) = \sum_{j=1}^{N} \sum_{i=1}^{q_j(n)} a_{ij}(n)\, \chi_{C_{ij}(n)}$$

be the projection of y onto the subspace of $L^2(X,\mu)$ spanned by $\{\chi_{C_{ij}(n)}: i = 1,\ldots,q_j(n);\ j = 1,\ldots,N\}$. Then

$$\begin{aligned}
& d^2(y(n),\ Z(\chi_{A_j(n)})) \\
& \le \Big\| y(n) - \Big(\sum_{i=1}^{q_j(n)} a_{ij}(n).\ \Pi_{h_{ij}(n)}\ \chi_{A_j(n)} \Big) \Big\|^2 \\
& = \Big\| y(n) - \Big(\sum_{i=1}^{q_j(n)} a_{ij}(n).\ \chi_{h_{ij}(n).A_j(n)} \Big) \Big\|^2 \\
& = \sum_{\substack{k=1 \\ k\neq j}}^{N} \Big(\sum_{i=1}^{q_k(n)} |a_{ik}(n)|^2\ \mu(C_{ik}(n)) \Big) \\
& \qquad + \sum_{i=1}^{q_j(n)} |a_{ij}(n)|^2\ \mu(C_{ij}(n) \backslash h_{ij}(n).A_j(n)) \\
& \le \sum_{\substack{k=1 \\ k\neq j}}^{N} \Big(\sum_{i=1}^{q_k(n)} |a_{ik}(n)|^2\ \mu(C_{ik}(n)) \Big) \\
& \qquad + (\theta/2). \sum_{i=1}^{q_j(n)} |a_{ij}(n)|^2\ \mu(C_{ij}(n))
\end{aligned}$$

(by (1)).

Summing over j, this gives

$$\sum_{j=1}^{N} d^2(y(n),\ Z(\chi_{A_j}(n)))$$

$$\leq (n - 1 + (\theta/2))\,\|y(n)\|^2$$

Inequality (2) now follows, because the hypothesis that $\xi(n) \to \varepsilon$ implies that $y(n) \to y$ as $n \to \infty$.

We are now in a positiion to use Proposition 1.1. Fix a positive integer k. Let us denote by m_k the essential supremum of the restriction to $\hat{G}_k$ of the spectral multiplicity function m. Choose an arbitrary positive integer ℓ no greater than m_k. Then, by 1.1(b), there exist unit vectors $y_1,\ldots,y_\ell$ in $L^2(X,\mu)$ such that

$$\sum_{s=1}^{\ell} d^2(y_s,Z(z)) \geq \ell-k, \text{ for all } z \in L^2(X,\mu).$$

Applying this inequality with $z = \chi_{A_j(n)}$, and summing over j, gives

$$\sum_{s=1}^{\ell} \sum_{j=1}^{N} d^2(y_s,Z(\chi_{A_j}(n))) \geq N(\ell-k).$$

Inequality (2) now implies that

$$\ell(N-1+(\theta/2)) \geq N(\ell-k).$$

When θ is less than 2, this is equivalent to:

$$\ell \leq 2Nk/(2-\theta).$$

Since ℓ was defined to be an arbitrary positive integer less than or equal to m_k, we conclude that

$$m_k \leq 2Nk/(2-\theta), \quad \text{for all } k \in \{1,2,\ldots\}.$$

This completes the proof.

Remark 3.2.

When G is abelian, so that dim $\lambda = 1$ for all λ in $\hat{G}$, Theorem 3.1 gives the uniform bound $[2N/(2-\theta)]$ on the spectral multiplicity of the G-action. If θ is less than $2/(N+1)$, this bound is equal to N, the best that could be expected using multiplicity N approximations.

Chacon [2] obtained the bound N for the special case (see Definition 2.5') of a multiplicity N approximation, with speed θ/n, $0 \leq \theta < 2/(N+1)$, of a measure-preserving automorphism of a probability space. Stepin [23] showed that a measure-preserving automorphism of a probability space which admits a cyclic approximation with speed θ/n, $\theta < 2$, must have spectral multiplicity uniformly less than or equal to $[2/(2-\theta)]$.

See Theorem 2.2 of the next chapter for another generalization, in a different direction, of these results of [2] and [23].

Remark 3.3.

Returning to the general case of an action of a type I group, observe that if one is given a multiplicity N approximation with speed $o(1/n)$, then the theorem implies that

$$m(\lambda) \leq N.\dim \lambda \quad , \quad \text{for almost all } \lambda \in \hat{G}.$$

By virtue of Proposition 1.1(a), this inequality may be interpreted as bounding by N the number of cyclic components needed to make up the unitary representation induced from the group action.

The following is a routine generalization of Theorem 1 of [23]. The proof is included because we wish to apply the more general version of this result in Chapter III.

<u>THEOREM 3.4</u>. [23]

Let there be given a measure (μ)-preserving action of a non-compact, locally compact, second countable, abelian group G on a standard Borel space X. Suppose that there exists a sequence of semi-partitions $\xi(n)$, $n = 1,2,\ldots$, and a sequence of group elements $g(1), g(2), \ldots$ satisfying

(i) the elements of each semi-partition $\xi(n)$ are all of finite measure,

(ii) $\xi(n) \to \varepsilon$, as $n \to \infty$,

(iii) $\lim_{n\to\infty} g(n) = \infty$, and

(iv) there exists a constant $\theta < 1$, independent of n, such that, whenever $C \in \xi(n)$,

$$\mu(g(n).C \,\Delta\, C) \leq \theta.\mu(C).$$

Then the maximal spectral type of the group action is singular with respect to the Haar measure on the dual group $\hat{G}$.

<u>Proof</u>

As before, let Π denote the unitary representation induced on $L^2(X,\mu)$ by the action of G on X.

If the maximal spectral type of Π were to have a component which was absolutely continuous with respect to the Haar measure on $\hat{G}$, then, by a general version of the Riemann-Lebesgue lemma, there would exist some non-zero vector y in $L^2(X,\mu)$ with $\lim_{g\to\infty} \langle \Pi_g y, y\rangle = 0$. The idea of the proof is to show that no such y can exist.

Let y be an arbitrary element of $L^2(X,\mu)$. Fix n, and let $y(n) = \sum_{C\in\xi(n)} a_C\chi_C$ be the projection of y onto the subspace of $L^2(X,\mu)$ spanned by $\{\chi_C : C \in \xi(n)\}$. Then

$$|\langle \Pi_{g(n)}y(n), \ y(n)\rangle|$$

$$= |\langle \sum_{C\in\xi(n)} a_C(\chi_{(g(n).C\setminus C)} + \chi_{(g(n).C \cap C)}),$$

$$\sum_{C\in\xi(n)} a_C(\chi_{(C\setminus g(n).C)} + \chi_{(C \cap g(n).C)})\rangle|$$

$$= \langle \sum_{C\in\xi(n)} a_C\, \chi_{g(n).C\setminus C}, \sum_{C\in\xi(n)} a_C\, \chi_{C\setminus g(n).C}\rangle$$

$$+ \sum_{C\in\xi(n)} |a_C|^2\, \mu(g(n).C \cap C)|$$

$$\geq \sum_{C\in\xi(n)} |a_C|^2\, \mu(g(n).C \cap C)$$

$$- \|\sum_{C\in\xi(n)} a_C\, \chi_{g(n).C\setminus C}\| \cdot \|\sum_{C\in\xi(n)} a_C\chi_{C\setminus g(n).C}\|$$

(by Schwarz's inequality)

$$= \sum_{C\in\xi(n)} |a_C|^2\, (\mu(g(n).C \cap C) - \mu(g(n).C\setminus C))$$

$$\geq (1-\theta) \sum_{C\in\xi(n)} |a_C|^2\, \mu(C) \qquad \text{(by (iv))}$$

$$= (1-\theta)\, \|y(n)\|^2 .$$

Since the above argument is valid for any n, and (ii) implies that $y(n) \to y$, as $n \to \infty$, it follows that

$$\liminf_{n \to \infty} \left| \left\langle \Pi_{g(n)} y, y \right\rangle \right| \geq (1-\theta) \|y\|^2 , \text{ for all } y \in L^2(X,\mu).$$

Since $g(n) \to \infty$, as $n \to \infty$, and $(1-\theta)$ is greater than zero, this completes the proof.

Remark 3.5.

The hypotheses of Theorem 3.4 are satisfied if the group action admits a cyclic approximation with speed θ/n, $0 \leq \theta < 1$, provided that the group elements $g_1(n), \ldots, g_{q(n)}(n)$, $n = 1,2,\ldots$, specified in Definition 2.6 may be chosen so that

$$\lim_{n\to\infty} g_{q(n)}(n) . g_{q(n)-1}(n) . \ \ldots \ . g_1(n) = \infty.$$

In particular, if a measure-preserving automorphism admits a cyclic approximation with speed θ/n, $0 \leq \theta < 1$, then it has singular spectrum (in this case, $G = Z$, and $g(n) = g_{q(n)}(n) + \ldots + g_1(n) = q(n)$, for all n). See Stepin [23] and Theorem II. 2.5.

The following theorem is a straightforward generalization of Theorem 2.1 of [12]. Its proof is included for the sake of completeness.

THEOREM 3.6. [12]

Suppose that a measure (μ)-preserving group action, $(g,x) \to g.x$, $g \in G$, $x \in X$, admits a cyclic approximation with speed θ/n, $\theta \geq 0$.

Then that action has a finite ergodic decomposition. If $\theta < 4$, then the action is ergodic. In general, the number of ergodic components is no more than max $(1, \theta/2)$.

Proof

Let $\xi(n) = \{C_1(n),\ldots,C_{q(n)}(n)\}$, $n = 1,2,\ldots$ and $g_1(n),\ldots,g_{q(n)}(n)$ be as specified by Definition 2.6.

Suppose that there exist N disjoint, non-null, G-invariant, Borel subsets $B_1,\ldots,B_N$ of X, where N is assumed to be at least two. Choose arbitrary $\delta > 0$. From the hypothesis that $\xi(n) \to \varepsilon$, as $n \to \infty$, it follows that there exists a positive integer n, depending on δ, such that integers $i_1 < i_2 < \ldots < i_N$ may be chosen from $\{1,\ldots,q(n)\}$ with

$$\mu(C_{i_k}(n) \cap B_k) \geq (1-\delta)\ \mu(C_1(n)), \text{ for } k = 1,\ldots,N. \qquad (1)$$

Note that a reordering of $B_1,\ldots,B_N$ may have been necessary to make this possible (for some such ordering, infinitely many integers n have the specified property).

Let us denote i_1 also by i_{N+1}.

For all $k = 1,\ldots,N$, it follows from (1) that

$$\mu(C_{i_{k+1}}(n) \cap B_k) < \delta\ \mu(C_1(n)),$$

and, taking into account the G-invariance of B_k, that

$$\mu(h.C_{i_k}(n) \cap B_k) \geq (1-\delta)\ \mu(C_1(n)), \text{ for all } h \in G.$$

Together, these inequalities imply that, for all $k = 1,\dots,N$,

$$\mu(h.C_{i_k}(n) \,\Delta\, C_{i_{k+1}}(n)) \geq (2-4\delta)\,\mu(C_1(n)),$$

$$\text{for any } h \in G. \tag{2}$$

On the other hand, if

$$h_k = \begin{cases} g_{i_{k+1}-1}(n).\, g_{i_{k+1}-2}(n).\, \dots\, .\, g_{i_k}(n), & \text{for } k \in \{1,\dots N-1\}, \\ g_{i_1-1}(n).\, \dots\, .\, g_1(n).g_{q(n)}(n).g_{q(n)-1}(n)\dots g_{i_N}(n), & \text{for } k=N, \end{cases}$$

then it follows from condition (iii) of Definition 2.6 that

$$\sum_{k=1}^{N} \mu(h_k.C_{i_k}(n) \,\Delta\, C_{i_{k+1}}(n)) \leq \theta\,\mu(C_1(n)). \tag{3}$$

Applying inequality (2) to the left side of (3) gives

$$N(2-4\delta)\,\mu(C_1(n)) \leq \theta\,\mu(C_1(n)).$$

Since δ was chosen arbitrarily, this implies that

$$N \leq \theta/2. \tag{4}$$

If $\theta < 4$, then (4) gives a contradiction to the assumed existence of more than one invariant set. Hence, in this case the action must be ergodic. For $\theta \geq 4$, inequality (4) gives the required bound on the number of ergodic components.

Remark 3.7.

Unlike the other results of this thesis, Theorem 3.6 applies to actions of arbitrary groups.

As a first application of the theorems of this chapter, observe that it has now been shown that the Z^2-action introduced as Example 2.9 is ergodic, and has simple, singular spectrum (note that these properties may also be obtained by other, direct means).

CHAPTER II

THE SPECTRAL PROPERTIES OF SKEW PRODUCT TRANSFORMATIONS

§1. Skew Products - Definitions and Preliminary Results

For the purposes of this chapter, T shall denote a measure-preserving automorphism of a standard Borel space X, where the measure μ on X is taken to be non-atomic and σ-finite. Also, let there be given a locally compact, second countable, abelian group A equipped with Haar measure da.

Definition 1.1. [22]

Let ϕ be a measurable function on X with values in A. Then the skew product A-extension of T determined by the function ϕ is the transformation $T_\phi : X \times A \to X \times A$ defined as follows

$$T_\phi (x,a) = (Tx, a+\phi(x)), \text{ for all } (x,a) \in X \times A.$$

With respect to the product measure $d\mu . da$, the skew product T_ϕ is a measure preserving automorphism of $X \times A$.

Definition 1.2. [22]

A measurable function $\phi : X \to A$ is said to be a coboundary if $\phi = \psi \cdot T - \psi$, for some measurable function $\psi: X \to A$. Two functions are said to be cohomologous if they differ by a coboundary.

PROPOSITION 1.3. [22]

Let ϕ and ϕ' be cohomologous measurable functions from X to A. Then the skew products T_ϕ and $T_{\phi'}$ are isomorphic measure-preserving automorphisms of $X \times A$.

Proof

Suppose that $\phi - \phi' = \psi \circ T - \psi$. Then $ST_{\phi'}S^{-1} = T_\phi$, where S is the measure-preserving automorphism of $X \times A$ given by:

$$S(x,a) = (x, a + \psi(x)), \text{ for all } (x,a) \in X \times A.$$

From now on, we shall denote the circle group $\{z \in \mathbb{C} : |z| = 1\}$ by K.

Definition 1.4.

If ρ is an arbitrary measurable function on X with values in the circle group K, then $V_{T,\rho}$ denotes the unitary operator defined on $L^2(X,\mu)$ as follows:

$$V_{T,\rho}\, y(x) = \rho(x).y(Tx), \text{ for all } y \in L^2(X,\mu),\ x \in X.$$

PROPOSITION 1.5.

If ρ and ρ' are cohomologous measurable functions on X with values in K, then the unitary operators $V_{T,\rho}$ and $V_{T,\rho'}$ are unitarily equivalent.

Proof

Suppose that $\rho/\rho' = \psi\circ T/\psi$, where $\psi : X \to K$ is measurable. Then $W^{-1}V_{T,\rho'}W = V_{T,\rho}$ where W is the unitary operator defined on $L^2(X,\mu)$ as follows:

$$Wy\,(x) = \psi(x).y(x), \text{ for all } y \in L^2(X,\mu) \text{ and } x \in X.$$

The next proposition shows that operators of the form $V_{T,\rho}$ arise naturally in the spectral analysis of skew product extensions of T.

Let T_ϕ be as in Definition 1.1 and let $\hat{A} = \{\gamma : \gamma$ is a continuous homomorphism from A to K$\}$ denote the dual of the abelian group A. Then if

$$U_{T_\phi} : L^2(X \times A) \to L^2(X \times A)$$
$$y \longmapsto y\circ T_\phi$$

is the unitary operator induced from T_ϕ, one has:

PROPOSITION 1.6.

The unitary operator U_{T_ϕ} is unitarily equivalent to $\int_{\hat{A}}^{\oplus} V_{T,\gamma\circ\phi}\, d\gamma$, the direct integral operator on the space $\int_{\hat{A}}^{\oplus} L^2(X,\mu)\, d\gamma$ which acts on each norm-square-integrable vector field $\gamma \to y_\gamma : \hat{A} \to L^2(X,\mu)$ as follows:

$$\left(\left(\int_{\hat{A}}^{\oplus} V_{T,\gamma\circ\phi}\, d\gamma\right).y\right)_\gamma = V_{T,\gamma\circ\phi}\, y_\gamma, \text{ for all } \gamma \in \hat{A}.$$

<u>Proof</u>

Let $W: L^2(X \times A) \to \int_{\hat{A}}^{\oplus} L^2(X,\mu)\, d\gamma$ be the Fourier transform in the second co-ordinate, defined as follows:

For all $y \in L^2(X \times A)$, Wy is the vector field which, in the fibre over $\gamma \in \hat{A}$, takes the value $(Wy)_\gamma \in L^2(X,\mu)$ given by setting

$$(Wy)_\gamma\ (x) = \int_A y(x,a)\ \overline{\gamma(a)}\ da, \quad \mu\text{-a.e. on } X,$$

where the integral converges in the L^2 sense.

For a suitable normalization of the Haar measures da and $d\gamma$, the Plancherel theorem implies that W is a Hilbert space isomorphism. Furthermore, given arbitrary $y \in L^2(X \times A)$, then, for μ-a.e. x in X, one has

$$\begin{aligned}
(WU_{T_\phi} y)_\gamma\ (x) &= \int_A y(Tx,\ a+\phi(x))\ \overline{\gamma(a)}\ da \\
&= \int_A y(Tx,a)\ \overline{\gamma(a-\phi(x))}\ da \\
&= \int_A y(Tx,a).\ \overline{\gamma(a)}.\ \overline{\gamma(-\phi(x))}\ da \\
&= (\gamma\circ\phi)(x).\ (Wy)_\gamma\ (Tx) \\
&= (V_{T,\gamma\circ\phi}\ (Wy)_\gamma)(x) \\
&= ((\int_{\hat{A}}^{\oplus} V_{T,\gamma\circ\phi}\ d\gamma)\ Wy)_\gamma(x)
\end{aligned}$$

Thus,

$$WU_{T_\phi} = (\int_{\hat{A}}^{\oplus} V_{T,\gamma\circ\phi}\ d\gamma)W$$

This establishes the desired unitary equivalence.

§2. Approximations and the Spectral Properties of Skew Products

With Proposition 1.6 as motivation, let us see how the method of approximations may be adapted to the determination of the spectral properties of the operators $V_{T,\rho}$ of Definition 1.4.

It will be necessary to restrict our attention to measurable functions $\rho : X \to K$ which take only countably many values. Each such function determines a partition

$$\eta = \{\rho^{-1}(z) : z \in \rho(X)\}$$

of the space X.

Recall Definitions I. 2.5' and I. 2.6', and observe that the semi-partitions $\xi(n)$, $n = 1,2,...$ which occur in an approximation of a measure-preserving automorphism are always summable, in the sense that, for each of them, the sum of the measures of its elements is finite.

Given any summable semi-partition ξ of X, a probability measure $\mu_\xi = \mu(\,.\,|\bigcup_{C\in\xi} C)$ may be defined on X as follows:

$$\mu_\xi(B) = \frac{\mu(B \cap (\bigcup_{C\in\xi} C))}{\mu(\bigcup_{C\in\xi} C)}$$

, for all measurable $B \subset X$.

In the following definition $f(n)$, $n = 1,2,...$ denotes a sequence of non-negative real numbers.

Definition 2.1. (c.f.Goodson [7])

A semi-partition η is said to be approximated with speed $f(n)$ by a sequence of finite summable semi-partitions $\xi(n)$, $n = 1,2,...$ if

$$\mu_{\xi(n)}\left(\bigcup_{B\in\eta} B \,\Delta\, B(\xi(n))\right) \leq f(q(n)), \text{ for all } n,$$

where each $B(\xi(n))$ is as defined by I. 2.2, and $q(n)$ denotes the number of elements in $\xi(n)$.

The next result generalizes Lemma 2 of Goodson [7] (see also Oseledec [19] and Stepin [23]). The proof relies heavily on the arguments used by Goodson, combining them with those used to prove Theorem I. 3.1.

THEOREM 2.2.

Let η be the partition of X determined by a measurable function $\rho: X \to K$ taking only countably many values. Suppose that the measure-preserving automorphism T admits a multiplicity N approximation with speed θ/n, $\theta \geq 0$.
Let

$$\xi(n) = \xi_1(n) \vee \xi_2(n) \vee \dots \vee \xi_N(n)$$
$$= \{C_{ij}(n) : i=1,...,q_j(n);\ j = 1,...,N\},\ n = 1,2,...$$

be the sequence of semi-partitions defining this approximation (see Definition I. 2.5'). Suppose that, for each $j = 1,...,N$, the partition η is approximated by the sequence of j^{th} components of $\xi(n)$,

$$\xi_j(n) = \{C_{ij}(n) : i = 1,...,q_j(n)\},\ n = 1,2,....\ ,$$

with speed δ/n, $\delta \geq 0$.

Then, if $\theta + 2\delta < 2$, the spectral multiplicity of the unitary operator $V_{T,\rho}$ cannot exceed $[\ 2N/(2 - (\theta + 2\delta))\ ]$.

Proof

For each $B \in \eta$, let ρ_B denote the constant value taken on B by the function ρ. Fixing, for the moment, $j \in \{1,\ldots,N\}$ and $n \in \{1,2,\ldots\}$, let $\rho_n^{(j)} : X \to K$ be the measurable function defined as follows:

$$\rho_n^{(j)}(x) = \begin{cases} \rho_B, & \text{if } x \in B(\xi_j(n)), \quad \text{for some } B \in \eta, \\ 1, & \text{if } x \in (\bigcup_{C \in \xi_j(n)} C) \setminus (\bigcup_{B \in \eta} B(\xi_j(n))), \\ \rho(x), & \text{if } x \in X \setminus (\bigcup_{C \in \xi_j(n)} C). \end{cases}$$

Note that Definition I. 2.2 guarantees that for distinct B and B' in η, $B(\xi_j(n))$ and $B'(\xi_j(n))$ are mutually disjoint unions of the elements of $\xi_j(n)$. Hence $\rho_n^{(j)}$ is a well-defined function, constant on each element of the semi-partition $\xi_j(n)$. Furthermore,

$$\{x \in X : \rho_n^{(j)}(x) \neq \rho(x)\}$$

$$\subset (\bigcup_{B \in \eta} B \,\Delta\, B(\xi_j(n))) \cap (\bigcup_{C \in \xi_j(n)} C)\ .$$

Since, by hypothesis, η is approximated by the sequence of semi-partitions $\xi_j(n)$, $n = 1,2,\ldots$ with speed δ/n, it follows from the above inclusion and Definition 2.1. that

$$\frac{\mu\{x \in X : \rho_n^{(j)}(x) \neq \rho(x)\}}{\mu(\bigcup_{C \in \xi_j(n)} C)} \leq \delta/q_j(n),$$

for all j and n. Since the $q_j(n)$ elements of $\xi_j(n)$ are, by Definition I. 2.5', all of the same measure, this inequality reduces to:

$$\mu\{x \in X : \rho_n^{(j)}(x) \neq \rho(x)\} \leq \delta.\mu(C_{1j}(n)). \tag{1}$$

Now, put

$$A_j(n) = \bigcap_{i=1}^{q_j(n)} T^{-(i-1)} C_{ij}(n).$$

Then, as in Theorem I. 3.1, the hypothesis that T be approximated with speed θ/n implies that

$$\mu(C_{ij}(n)\backslash T^{(i-1)}A_j(n)) \leq (\theta/2).\mu(C_{ij}(n)),$$

$$\text{for all } i = 1,\ldots,q_j(n). \tag{2}$$

Note that, by definition, $T^{(i-1)}A_j(n) \subset C_{ij}(n)$, for all $i = 1,\ldots,q_j(n)$. For the present proof, the choice of $A_j(n)$ needs to be modified. Following [19] and [7], we let

$$A_j'(n) = A_j(n) \cap (\bigcap_{i=1}^{q_j(n)} T^{-(i-1)} \{x \in X: \rho_n^{(j)}(x) = \rho(x)\}).$$

Then

$$\mu(A_j(n)) - \mu(A_j'(n))$$
$$= \mu(A_j(n) \setminus (\bigcap_{i=1}^{q_j(n)} T^{-(i-1)}\{x \in X : \rho_n^{(j)}(x) = \rho(x)\})),$$

$$= \mu\left(\bigsqcup_{i=1}^{q_j(n)} (A_j(n) \cap T^{-(i-1)} \{x \in X: \rho_n^{(j)}(x) \neq \rho(x)\})\right)$$

$$\leq \sum_{i=1}^{q_j(n)} \mu(T^{i-1} A_j(n) \cap \{x \in X : \rho_n^{(j)}(x) \neq \rho(x)\})$$

$$\leq \mu\{x \in X: \rho_n^{(j)}(x) \neq \rho(x)\},$$

where, to obtain the final inequality, we have used the fact that the sets $T^{i-1} A_j(n)$, $i = 1,\ldots,q_j(n)$, are mutually disjoint. From (1) and (2), we may now conclude that

$$\mu(C_{ij}(n) \setminus T^{(i-1)} A_j'(n))$$

$$< ((\theta + 2\delta)/2)\, \mu(C_{ij}(n)),$$

$$\text{for all } i = 1,\ldots,q_j(n). \tag{3}$$

Now, consider the $V_{T,\rho}$-cyclic subspace of $L^2(X,\mu)$ generated by the characteristic function of the set $A_j'(n)$. This subspace, $Z(\chi_{A_j'(n)})$, contains $\{V_{T,\rho}^{-(i-1)}\chi_{A_j'(n)} : i \in Z\}$. Let $i \in \{2,\ldots,q_j(n)\}$. Then

$$V_{T,\rho}^{-(i-1)}\, \chi_{A_j'(n)} = (1/\rho\circ T^{-1})(1/\rho\circ T^{-2})\ldots (1/\rho\circ T^{-(i-1)})\chi_{T^{i-1}A_j'(n)}$$

and, by the definition of $A_j'(n)$,

$$T^{i-1}A_j'(n) = T^{i-1}\left(\bigcap_{\ell=1}^{q_j(n)} T^{-(\ell-1)}(C_{\ell,j}(n) \cap \{x:\rho(x) = \rho_n^{(j)}(x)\})\right)$$

$$\subset \bigcap_{\ell=1}^{i-1} T^{i-\ell}\, (C_{\ell,j}(n) \cap \{x:\rho(x) = \rho_n^{(j)}(x)\})$$

$$= \bigcap_{\ell=1}^{i-1} T^{\ell}\, (C_{i-\ell,j}(n) \cap \{x: \rho(x) = \rho_n^{(j)}(x)\})$$

Since, for each $\ell \in \{1,\ldots, i-1\}$, the constancy of the function $\rho_n^{(j)}$ on the elements of $\xi_j(n)$ implies that $(1/\rho \circ T^{-\ell})$ is constant on

$$T^{\ell}\ (C_{i-\ell,j}(n) \cap \{x \in X : \rho(x) = \rho_n^{(j)}(x)\}),$$

it follows from the last inclusion above that the functions $(1/\rho \circ T^{-1}), (1/\rho \circ T^{-2}),\ldots, (1/\rho \circ T^{-(i-1)})$ are all constant on the set $T^{i-1} A'_j(n)$. Hence, $V_{T,\rho}^{-(i-1)}\ \chi_{A'_j(n)}$ is just a constant multiple of the characteristic function of $T^{i-1} A'_j(n)$. Since i was chosen arbitrarily from $\{2,\ldots,q_j(n)\}$, we conclude that

$$Z(\chi_{A'_j(n)}) \supset \text{lin.span}\ \{\chi_{A'_j(n)}, \chi_{TA'_j(n)}, \ldots$$
$$\ldots, \chi_{T^{q_j(n)-1}A'_j(n)}\}. \qquad (4)$$

To complete the proof, it only remains to apply once again the method of Theorem I. 3.1:

(a) From (3) and (4) and the fact that $\xi(n) \to \varepsilon$, as $n \to \infty$, it follows that

$$\limsup_{n\to\infty} \sum_{j=1}^{N} d^2(y, Z(A'_j(n)))$$
$$\leq (N-1 + (\theta+2\delta)/2)\ \|y\|^2,$$
$$\text{for all } y \in L^2(X,\mu).$$

(b) Using the inequality of (a), the result follows, as in I. 3.1, from Proposition I. 1.1.

Remark 2.3.

Modulo his restriction to probability spaces, Goodson [7] proved the above result in the case of multiplicity one approximations, i.e. for $N = 1$. In turn, this result contained as a special case a result of Oseledec [19], who considered only cyclic approximations, with X a probability space, and showed that in these circumstances, if $\theta + 2\delta < 1$, then the operator $V_{T,\rho}$ has simple spectrum (note that, in general, if $\theta + 2\delta < 2/(N+1)$, then Theorem 2.2 implies that the spectral multiplicity of $V_{T,\rho}$ is less than or equal to N).

Remark 2.4.

Letting the function ρ be identically equal to 1, one obtains as a special case of the above theorem the result that if a measure-preserving automorphism admits a multiplicity N approximation with speed θ/n, $0 \le \theta < 2$, then its spectral multiplicity cannot exceed $[2N/(2-\theta)]$. This, a generalization of the corresponding results of Chacon [2] and Stepin [23], also follows from I. 3.1 (see Remark I. 3.2).

Stepin [23] has shown that if a measure-preserving automorphism of a probability space admits a cyclic approximation with speed θ/n, for some $0 \le \theta < 1$, then it has singular spectrum (see also I. 3.5). The proof of the following theorem shows that Stepin's methods may be extended so as to apply to operators of the form $V_{T,\rho}$.

THEOREM 2.5.

Let η be the partition of X determined by a measurable function $\rho: X \to K$ taking only countably many values. Suppose that the measure-preserving automorphism T admits a cyclic approximation with speed θ/n, $\theta \geq 0$; and that η is approximated with speed δ/n, $\delta \geq 0$, by the sequence of semi-partitions

$$\xi(n) = \{C_i(n) : i=1,\ldots,q(n)\},\ n = 1,2,\ldots$$

specified by Definition I. 2.6'.

Then, if $\theta + 2\delta < 1$, the operator $V_{T,\rho}$ has singular spectral type.

Proof

As in I. 3.4, it will be sufficient to show that

$$\liminf_{n \to \infty} \left| \left\langle V_{T,\rho}^{\ q(n)} y, y \right\rangle \right| > 0, \quad \text{for all } y \in L^2(X,\mu).$$

By the construction used in 2.2, the fact that η is approximated with speed δ/n implies that there exists a sequence of measurable functions $\rho_n: X \to K$, $n = 1,2,\ldots$, with each ρ_n constant on the elements of $\xi(n)$, and

$$\mu\{x \in X: \rho_n(x) \neq \rho(x)\} \leq \delta\mu(C_1(n)),\ \text{for all } n.$$

Given $i \in Z$, it will be convenient to let $C_i(n)$ denote the unique element $C_{i'}(n)$ of $\xi(n)$ with $i' \equiv i \pmod{q(n)}$. Thus, $C_{q(n)+1}(n) = C_1(n)$, $C_{q(n)+2}(n) = C_2(n)$, etc.

Now, for all $n \in \{1,2,\ldots\}$ and $i \in \{1,\ldots,q(n)\}$,

define

$$D_i(n) = \bigcap_{k=0}^{q(n)} T^{-k}\, C_{i+k}(n),$$

and let

$$E_i(n) = D_i(n) \cap (\bigcap_{k=0}^{q(n)-1} T^{-k}\, \{x \in X\colon \rho_n(x) = \rho(x)\}).$$

Note that $T^k\, E_i(n) \subset T^k\, D_i(n) \subset C_{i+k}(n)$, for all $k \in \{0,\ldots,q(n)\}$. Also,

$$\mu(C_i(n)) - \mu(D_i(n))$$

$$\leq \sum_{k=1}^{q(n)} \mu(T^{-(k-1)}C_{i+k-1}(n) \setminus T^{-k}\, C_{i+k}(n))$$

$$= \frac{1}{2} \sum_{k=1}^{q(n)} \mu(TC_k(n) \,\Delta\, C_{k+1}(n))$$

$$\leq (\theta/2)\, \mu(C_i(n)) \qquad \text{(by Definition I. 2.6').}$$

Furthermore, since the sets $T^k D_i(n)$, $k = 0,\ldots,q(n)-1$, are mutually disjoint, we have

$$u(D_i(n)) - \mu(E_i(n))$$

$$= \mu(\bigcup_{k=0}^{q(n)-1} (D_i(n) \cap T^{-k}\, \{x \in X\colon \rho(x) \neq \rho_n(x)\}))$$

$$\leq \sum_{k=0}^{q(n)-1} (T^k D_i(n) \cap \{x \in X\colon \rho(x) \neq \rho_n(x)\})$$

$$\leq \mu\{x \in X\colon \rho(x) \neq \rho_n(x)\}$$

$$\leq \delta\, \mu(C_i(n)).$$

Thus,

$$\mu(C_i(n)) - \mu(E_i(n))$$

$$= \mu(C_i(n)) - \mu(D_i(n)) + \mu(D_i(n)) - \mu(E_i(n))$$

$$\leq ((\theta + 2\delta)/2).\mu(C_i(n)), \text{ for all } n = 1,2,\ldots,$$

$$\text{and } i = 1,\ldots,q(n). \qquad (1)$$

Now, for arbitrary i and n,

$$V_{T,\rho}^{-q(n)} \chi_{E_i(n)} = (1/\rho\circ T^{-1})(1/\rho\circ T^{-2})\ldots(1/\rho\circ T^{-q(n)})\chi_{T^{q(n)}E_i(n)},$$

and

$$T^{q(n)}E_i(n) = C_i(n) \cap (\bigcap_{k=0}^{q(n)-1} T^{q(n)-k}(C_{i+k}(n) \cap \{x:\rho(x) = \rho_n(x)\}))$$

$$= C_i(n) \cap (\bigcap_{k=1}^{q(n)} T^k(C_{i+q(n)-k}(n) \cap \{x:\rho(x) = \rho_n(x)\})),$$

where, in the last line, we have merely substituted $q(n) - k$ for k. Thus, since each of the functions $\rho\circ T^{-k}$, $k \in \{1,\ldots,q(n)\}$, takes the same constant value on the set $T^k(C_{i+q(n)-k}(n) \cap \{x:\rho(x) = \rho_n(x)\})$ as that taken by ρ_n on $C_{i+q(n)-k}(n)$, it follows that if $c_n \in K$ denotes the product of the $q(n)$ values taken by ρ_n on the elements of $\xi(n)$, then

$$V_{T,\rho}^{-q(n)} \chi_{E_i(n)} = (1/c_n)\, \chi_{T^{q(n)}E_i(n)},$$

$$\text{for all } n = 1,2,\ldots \text{ and } i = 1,\ldots,q(n). \qquad (2)$$

Now, suppose that y is an arbitrary element of $L^2(X,\mu)$. Then, for each $n \in \{1,2,\ldots\}$, let

$$y_n = \sum_{i=1}^{q(n)} a_i(n)\, \chi_{C_i(n)}$$

be the projection of y onto the subspace of $L^2(X,\mu)$ spanned by $\{\chi_C : C \in \xi(n)\}$, and put

$$y'_n = \sum_{i=1}^{q(n)} a_i(n)\, \chi_{E_i(n)}.$$

Fix n for the moment. From (2), one obtains

$$c_n\, V_{T,\rho}^{-q(n)}\, y'_n = \sum_{i=1}^{q(n)} a_i(n)\, \chi_{T^{q(n)}E_i(n)}.$$

Since $T^{q(n)}E_i(n) \subset C_i(n)$, for all $i \in \{1,\ldots,q(n)\}$,

it follows that the functions $(y_n - c_n\, V_{T,\rho}^{-q(n)} y'_n)$ and $c_n\, V_{T,\rho}^{-q(n)} y'_n$ are orthogonal. Similarly the inclusions $E_i(n) \subset C_i(n)$, for $i \in \{1,\ldots,q(n)\}$, imply that $(y_n - y'_n)$ and y'_n are orthogonal. Thus, by Pythagoras,

$$\begin{aligned} \|y_n - c_n\, V_{T,\rho}^{-q(n)}\, y'_n\|^2 &= \|y_n\|^2 - \|c_n\, V_{T,\rho}^{-q(n)}\, y'_n\|^2 \\ &= \|y_n\|^2 - \|y'_n\|^2 \\ &= \|y_n - y'_n\|^2 . \end{aligned} \tag{3}$$

Collecting these facts together, one obtains:

$$|\langle V_{T,\rho}^{q(n)}\, y_n,\, y_n \rangle|$$

$$= |\langle y_n,\, c_n V_{T,\rho}^{-q(n)}\, y_n \rangle|$$

$$= |\langle (y_n - c_n V_{T,\rho}^{-q(n)}\, y_n') + c_n V_{T,\rho}^{-q(n)}\, y_n',$$

$$c_n V_{T,\rho}^{-q(n)}(y_n - y_n') + c_n V_{T,\rho}^{-q(n)} y_n' \rangle|$$

$$= |\langle (y_n - c_n V_{T,\rho}^{-q(n)}\, y_n'),\, c_n V_{T,\rho}^{-q(n)}(y_n - y_n') \rangle$$

$$+ \|y_n'\|^2 \quad |$$

$$\geq \|y_n'\|^2 - \|y_n - y_n'\|^2 \qquad \text{(by (3) and Schwarz's inequality)}$$

$$= \sum_{i=1}^{q(n)} |a_i(n)|^2\, \mu(E_i(n)) - \sum_{i=1}^{q(n)} |a_i(n)|^2 \mu(C_i(n) \setminus E_i(n))$$

$$= \sum_{i=1}^{q(n)} |a_i(n)|^2\, (2\mu(E_i(n)) - \mu(C_i(n)))$$

$$\geq (1 - (\theta + 2\delta))\, \|y_n\|^2 \qquad \text{(by (1))}.$$

Since $\xi(n) \to \varepsilon$ as $n \to \infty$, the above implies that

$$\liminf_{n\to\infty} |\langle V_{T,\rho}^{q(n)}\, y, y \rangle|$$

$$\geq (1 - (\theta + 2\delta))\, \|y\|^2,$$

$$\text{for all } y \in L^2(X,\mu).$$

Since, by hypothesis, $(1 - (\theta+2\delta))$ is strictly positive, this completes the proof.

Remark 2.6.

By considering the case when the function ρ is identically equal to 1, one may recover from the above theorem its predecessor, Theorem 1 of Stepin [23], minus the restriction to probability spaces.

§3. Approximations and Essential Values - The Ergodicity of Skew Products

Let A be as introduced in §1. Assume that the topology on A is given by a metric d.

Definition 3.1. [22]

Let ϕ be a measurable function on X with values in A. Then an element a of the group A is said to be an essential value of ϕ (with respect to T) if either a = 0, or, for every $\varepsilon > 0$, and every non-null measurable set $B \subset X$, there exists a positive integer k such that

$$\mu(B \cap T^{-k}B \cap \{x \in X : d(\sum_{j=0}^{k-1} \phi(T^j x), a) < \varepsilon\}) > 0.$$

The set of essential values, with respect to T, of the function ϕ shall be denoted $E_T(\phi)$.

The importance of Definition 3.1 is demonstrated by the following proposition, which collects together a number of results proved in Sections 3 and 5 of Schmidt [22].

PROPOSITION 3.2. [22].

Let $\phi : X \to A$ be a measurable function. Then

(a) $E_T(\phi)$ is a closed subgroup of A,

(b) if $\phi' : X \to A$ is cohomologous to ϕ, then $E_T(\phi) = E_T(\phi')$,

(c) the skew product T_ϕ is ergodic if and only if $E_T(\phi) = A$,

(d) if the quotient topological group $A/E_T(\phi)$ is compact, then ϕ is cohomologous to a function taking values only in $E_T(\phi)$.

The following proposition gives a method of attack on the difficult problem of computing essential values. This method is closely related with that of Lemma 3.2.7 of Atkinson [1].

PROPOSITION 3.3.

Let $\phi : X \to A$ be measurable, and let $a \in A$. Suppose that there exists a sequence of semi-partitions $\xi(n)$, $n = 1,2,\ldots$ such that

(i) $\xi(n) \to \varepsilon$, as $n \to \infty$,

and

(ii) for any $\varepsilon > 0$, there exists a constant $\theta_\varepsilon > 0$, with the property that, for all sufficiently large n, every element C of $\xi(n)$ satisfies the following condition:

$$(*) \quad \begin{cases} \text{there exists a positive integer } k \text{ with} \\ \mu(C \cap T^{-k}C \cap \{x \in X:\ d(\sum_{j=0}^{k-1} \phi(T^j x), a) < \varepsilon\}) \\ \qquad \geq \theta_\varepsilon\, \mu(C). \end{cases}$$

Then $a \in E_T(\phi)$.

Proof

Let B be an arbitrary measurable subset of X with $0 < \mu(B) < \infty$. Choose $\varepsilon > 0$.

Then, for all but finitely many positive integers n, there must exist at least one element C of $\xi(n)$ such that

$$(**) \qquad \mu(C \setminus B) \le (\theta_\varepsilon/3)\, \mu(C).$$

This must be so, since, otherwise, in the notation of I.2.2, one would have $\mu(B \,\Delta\, B(\xi(n)))$ greater than $(\theta_\varepsilon/3)\, \mu(B(\xi(n)))$, for infinitely many n, clearly contradicting the hypothesis that $\xi(n) \to \varepsilon$, as $n \to \infty$.

Thus, (i) and (ii) imply that, for all but finitely many values of n, there exists a member C of $\xi(n)$ satisfying both (*) and (**). For such C, letting k denote the positive integer specified in (*), one has

$$\mu(B \cap T^{-k}B \cap \{x \in X : d(\sum_{j=0}^{k-1} \phi(T^j x), a) < \varepsilon\})$$

$$\ge \mu((B \cap C) \cap T^{-k}(B \cap C) \cap \{x : d(\sum_{j=0}^{k-1} \phi(T^j x), a) < \varepsilon\})$$

$$\ge \mu(C \cap T^{-k}C \cap \{x : d(\sum_{j=0}^{k-1} \phi(T^j x), a) < \varepsilon\})$$

$$- \mu(C \setminus B) - \mu(T^{-k}(C \setminus B))$$

$$\ge (\theta_\varepsilon/3)\, \mu(C)$$

$$> 0.$$

Since ε was chosen arbitrarily, and since in Definition 3.1 it is clearly sufficient to consider only those B of finite measure, the proof is complete.

Remark 3.4.

It is clear that Proposition 3.3 would still hold if hypothesis (i) were weakened and the sequence of semi-partitions was only assumed to be dense, in the sense of Halmos [8], §41. In other words, it would be sufficient to have $\liminf_{n \to \infty} \mu(B \,\Delta\, B(\xi(n))) = 0$, for all measurable $B \subset X$ with $\mu(B) < \infty$.

§4. Finite Skew Products - the Discrete Part of the Spectrum

If the group A introduced in §1 is chosen to be Z_r, the group of r^{th} rooths of unity, $r \in \{2,3,\ldots\}$, then the direct integral decomposition given by Proposition 1.6 takes a particularly simple form:

Given a measurable function $\phi: X \to Z_r$, the unitary operator induced from the skew product $T_\phi : X \times Z_r \to X \times Z_r$ admits the direct sum decomposition

$$U_{T_\phi} \simeq \bigoplus_{i=0}^{r-1} V_{T,\phi^i} \cdot$$

Note that the direct summand $V_{T,1}$, corresponding to the value zero of the index i, is just the unitary operator

$$U_T : L^2(X,\mu) \to L^2(X,\mu)$$
$$f \mapsto f \circ T$$

induced from the measure-preserving automorphism T.

An attempt to use the above decomposition as a means of working out the spectral properties of finite skew product extensions of T immediately leads to the problems of

(a) determining the spectral properties of operators of the form $V_{T,\rho}$ of Definition 1.4, and

(b) comparing the spectral properties of $V_{T,\rho}$ for different choices of $\rho : X \to K$.

In general, these problems are difficult. However, if one focusses on the discrete part of the spectrum, then some simple facts emerge. For the purposes of this section, it will be assumed that μ is finite, and that T is ergodic, with group of eigenvalues $\Lambda \leq K$.

Fix $r \in \{2,3,\ldots\}$, and let $\Lambda^{1/r}$ denote the subgroup of K consisting of all possible r^{th} roots of the elements of Λ. Note that Λ is a subgroup of $\Lambda^{1/r}$. For convenience, let us identify each element of K with the corresponding constant function on X. If two measurable functions $\rho, \rho' : X \to K$ are cohomologous (see 1.2), then let us write $\rho \sim \rho'$. Note that $\sim$ is an equivalence relation and that, under pointwise multiplication of representatives, the set of cohomology classes is an abelian group.

PROPOSITION 4.1.

(a) Let ρ be a measurable function on X with values in K. Then the eigenvalues of the operator $V_{T,\rho}$ (if any exist) are precisely those elements of K which are cohomologous to ρ (when considered as constant functions on X).

(b) Let $\lambda \in K$. Then, in order that λ be an eigenvalue of $V_{T,\rho}$ for some measurable $\rho : X \to Z_r$, it is necessary and sufficient that λ belong to $\Lambda^{1/r}$.

(c) Let Δ denote the set of cohomology classes of those functions $\rho : X \to K$ for which $V_{T,\rho}$ possesses an eigenvalue.

Then the map

$$\rho \longmapsto \quad \text{the set of eigenvalues of } V_{T,\rho}$$

defines a group isomorphism

$$J: \Delta \xrightarrow[\text{onto}]{1.1} K/\Lambda$$

under which the subgroup of Δ consisting of those cohomology classes which contain functions taking their values only in Z_r is identified with $\Lambda^{1/r}/\Lambda$.

Proof

(a) Suppose that $\lambda \in K$ is an eigenvalue of $V_{T,\rho}$. Then $\rho.(y \circ T) = \lambda.y$, for some $y \in L^2(X,\mu)$. By taking absolute values and using the ergodicity of T, it follows that the eigenfunction y must be of almost-everywhere-constant modulus. Thus, y may be chosen so that $|y|$ is everywhere equal to 1, in which case the identity

$$\lambda/\rho \quad = \quad y \circ T/_y$$

means that $\lambda \sim \rho$.

The reverse implication is clear.

(b) If

$$\rho(y \circ T) = \lambda.y, \text{ for some } \rho: X \to Z_r,$$

and some $y \in L^2(X,\mu)$, then

$$y^r \circ T = \lambda^r.y^r.$$

By definition, this means that $\lambda \in \Lambda^{1/r}$.

On the other hand, assume that $\lambda \in \Lambda^{1/r}$. Then $\lambda^r \in \Lambda$, and there must exist a function $y : X \to K$ with

$$y \circ T = \lambda^r . y .$$

Choose any measurable r^{th} root $y^{1/r}$ of y. Then, since

$$(\lambda y^{1/r}/y^{1/r} \circ T)^r = (\lambda^r y/y \circ T) = 1,$$

it follows that the function

$$\rho = \lambda y^{1/r} / y^{1/r} \circ T$$

takes its values in Z_r. For this choice of ρ, λ is an eigenvalue of $V_{T,\rho}$.

(c) From (a) and Proposition 1.5, it follows that, for any measurable function $\rho : X \to K$, and any $\lambda \in K$, the following are equivalent:

(i) λ is an eigenvalue of $V_{T,\rho}$.

(ii) $\rho \sim \lambda$.

(iii) $V_{T,\rho}$ is unitarily equivalent to the operator λU_T.

(iv) $V_{T,\rho}$ has $\lambda\Lambda$ as its set of eigenvalues.

The equivalence of (i) and (ii) implies that Λ is just the group formed by the cohomology classes of the constant functions. Since a constant function $\rho \equiv \lambda'$, satisfies (iv) if and only if $\lambda'\Lambda = \lambda\Lambda$ (because $V_{T,\lambda'} = \lambda' U_T$), one may deduce from the equivalence of (ii) and (iv) that, as far as constant functions are concerned, cohomology classes are just (K/Λ)-cosets. The map J is simply the group isomorphism obtained by identifying each cohomology class in Δ with the (K/Λ)-coset formed by its constant elements.

The remaining assertion of (c) follows immediately from (b).

Remark 4.2.

If the measure-preserving transformation T has purely discrete spectrum, then it is clear from the proof of (c), above, that any operator of the form $V_{T,\rho}$ has either purely discrete or purely continuous spectral type; and that, unless $\rho : X \to K$ is a coboundary, this spectral type is disjoint from that of U_T.

Remark 4.3.

If the group of eigenvalues, Λ, is finitely generated, then $\Lambda^{1/r}/\Lambda$ is finite. In this case, it follows from (c), above, that if a function ρ takes its values in Z_r, then, unless this function belongs to one of only finitely many cohomology classes, the operator $V_{T,\rho}$ will have purely continuous spectrum.

Now, for all $\alpha \in (0,1)$, let T_α be the rotation of the group $\mathbb{R}/Z \cong [0,1)$ defined as follows:

$$T_\alpha x = x + \alpha \pmod 1 \quad , \quad \text{for all } x \in [0,1).$$

The next chapter will be devoted to a study of certain skew product extensions of the (Lebesgue) measure-preserving transformations T_α, $\alpha \in (0,1)$. The present chapter finishes with a characterization of those measurable functions $\rho : X \to K$ for which the operator $V_{T_\alpha,\rho}$ has discrete spectrum.

For all $x \in \mathbb{R}$, let $\langle\!\langle x \rangle\!\rangle$ denote the distance from x to the nearest integer. The following is a reinterpretation of Proposition 5 of Veech [24].

PROPOSITION 4.4.

For all $\alpha \in (0,1)$ and all measurable functions $\rho : X \to K$, the following are equivalent:

(a) The operator $V_{T_\alpha,\rho}$ has an eigenvalue (hence purely discrete spectrum).

(b) $$\lim_{\langle\!\langle n\alpha \rangle\!\rangle \to 0} |\langle (V_{T_\alpha,\rho})^n 1, 1 \rangle| = 1$$

Proof

Suppose that $V_{T_\alpha,\rho}$ has an eigenvalue. Then $\rho = \lambda.(y/y \circ T_\alpha)$, where $|\lambda| = 1$, and $y : [0,1) \to K$ is a measurable function. Hence,

$$(V_{T_\alpha,\rho})^n 1 = \lambda^n.(y/y \circ T_\alpha^n),$$

so that

$$|\langle (V_{T_\alpha,\rho})^n 1, 1 \rangle| = |\langle y, y \circ T_\alpha^n \rangle| ,$$

for all $n \in Z$. Hence,

$$\lim_{\langle\!\langle n\alpha \rangle\!\rangle \to 0} |\langle (V_{T_\alpha,\rho})^n 1, 1 \rangle| = \|y\|^2 = 1.$$

For the converse, note, as does Veech [24], that (b) is sufficient for

$$\{n \in Z: |\langle (V_{T_\alpha,\rho})^n 1, 1 \rangle| > \tfrac{1}{2}\}$$

to have positive density in the integers. From this it follows that

$$\lim_{n\to\infty} (1/(2n+1)) \sum_{j=-n}^{n} \left| \left\langle (V_{T_\alpha,\rho})^j 1, 1 \right\rangle \right|^2$$

$$> 0.$$

Hence, the spectral measure of the vector $1 \in L^2[0,1)$ has an atom. Such atoms are eigenvalues of $V_{T_\alpha,\rho}$.

CHAPTER III

APPLICATIONS - SKEW PRODUCTS OVER IRRATIONAL ROTATIONS

§1. Preliminaries - Diophantine Approximations

This chapter deals with skew products based on the rotations, T_α, $\alpha \in (0,1)$, introduced at the end of II. 4. In applying the method of approximations to the examples to be considered, rational approximations of the parameter α (and other parameters to be introduced later) will play a key role. Before stating the results which will guarantee the existence of these approximations, let us, for ease of reference, list some definitions: For $\alpha \in (0,1)$,

$$T_\alpha x = x + \alpha \pmod 1, \quad \text{for all } x \in [0,1).$$

For all $x \in \mathbb{R}$,

$[x]$ denotes the integer part of x,

$\langle x\rangle = x - [x]$, the fractional part of x,

and $\langle\langle x\rangle\rangle = \min(\langle x\rangle, 1 - \langle x\rangle)$, the distance from x to the nearest integer.

The following is a basic result of the theory of continued fractions (see page 31 of Khinchin [13]).

PROPOSITION 1.1.

Given any irrational $\alpha \in (0,1)$, there exists a sequence of irreducible fractions p_n/q_n, $n = 1,2,\ldots$ satisfying

(i) $q_n \nearrow \infty$, as $n \to \infty$,

and

(ii) $|\alpha - p_n/q_n| < 1/(\sqrt{5}\, q_n^2)$, for all n.

The next result, to be used in §3, is an obvious consequence of Theorem 1 of Hartman and Szüsz [9].

PROPOSITION 1.2.

Let $a > 0$ and b be integers. Then, for almost every $\alpha \in (0,1)$, there exists a sequence of irreducible fractions p_n/q_n, $n = 1,2,\ldots$ such that

(i) $q_n \nearrow \infty$, as $n \to \infty$,

(ii) $q_n^2 \, |\alpha - p_n/q_n| \to 0$, as $n \to \infty$,

and

(iii) $q_n \equiv b \pmod a$, for all n.

PROPOSITION 1.3.

Let $q_1, q_2, \ldots$ be an arbitrary strictly increasing sequence of positive integers, and let f be a non-negative function defined on the set of positive integers. Suppose that $f(n) \to 0$, as $n \to \infty$.

Then, given any $t \in [0,1]$, those real numbers $\beta \in (0,1)$ for which there exist subsequences, q_{n_k}, $k = 1,2,\ldots$, of the given sequence of positive integers such that

(i) $\langle q_{n_k} \beta \rangle \to t$, as $k \to \infty$.

and

(ii) $|\langle q_{n_k} \beta \rangle - t| > f(q_{n_k})$, for all k,

form a residual subset of full measure in $(0,1)$.

The "full measure" part of this proposition follows from a classical result of Weyl [25], asserting that, under the given conditions, the sequence $q_n\beta$, $n = 1,2,\ldots$ is uniformly distributed, mod 1, for almost all $\beta \in \mathbb{R}$. In an appendix, we shall present another proof - from first principles.

§2. A Class of Z_2-Extensions of Irrational Rotations

Fix irrational $\alpha \in (0,1)$ and let p_n/q_n, $n = 1,2,\ldots$ be the sequence of rational approximations of α specified by Proposition 1.1. Then, following Katok and Stepin [12], a cyclic approximation of the irrational rotation T_α may be constructed as follows:

For each $n \in \{1,2,\ldots\}$, define a partition

$$\xi(n) = \{C_i(n) : i = 1,\ldots,q_n\}$$

by letting

$$C_i(n) = T^{i-1}_{p_n/q_n} [0,1/q_n), \text{ for } i = 1,\ldots,q_n.$$

That the sets $C_1(n),\ldots,C_{q_n}(n)$ are mutually disjoint follows from the irreducibility of the fraction p_n/q_n. In fact, $\xi(n)$ is just a reordering of the partition $\{[0,1/q_n), [1/q_n, 2/q_n), [(q_n-1)/q_n,1)\}$. Furthermore, if μ denotes the Lebesgue measure on $[0,1)$, then

$$\begin{aligned}(1/(q_n\ \mu(C_1(n)))\ )\ \sum_{i=1}^{q_n} \mu(T_\alpha C_i(n) \,\Delta\, C_{i+1}(n)) &= q_n.2|\alpha - p_n/q_n| \\ &< 2/(\sqrt{5}\ q_n) \quad \text{(by 1.1).}\end{aligned}$$

The sequence of partitions $\xi(n)$, $n = 1,2,\ldots$ clearly satisfies conditions (i) and (ii) of Definition I. 2.6'. Thus, this sequence defines a cyclic approximation of T_α - with speed $2/(\sqrt{5}n)$.

Now, for each $\beta \in [0,1)$ define a function $\rho_\beta : [0,1) \to Z_2$ $(= \{1, -1\})$ as follows:

$$\rho_\beta(x) = \begin{cases} -1 & , \text{ if } 0 \le x < \beta\ , \\ 1 & , \text{ if } \beta \le x < 1. \end{cases}$$

PROPOSITION 2.1.

For any given irrational $\alpha \in (0,1)$, the operator V_{T_α,ρ_β} (see Definition II. 1.4) has simple, singular spectrum for all values of β in some residual, measure-one subset of $[0,1)$.

Proof

Let η denote the partition of $[0,1)$ determined by the function ρ_β , i.e. let $\eta = \{[0,\beta), [\beta,1)\}$. Choose some positive real number δ less than $\frac{1}{2}$.

If β satisfies the inequality $\langle\langle q_n \beta\rangle\rangle \leq \delta$ for infinitely many values of n, then η will be approximated with speed δ/n (see Definition II. 2.1) by some subsequence of the sequence of partitions $\xi(n) = \{[0,1/q_n),\ldots,[(q_n-1)/q_n,1)\}$, $n = 1,2,\ldots$. For such β , if $(2/\sqrt{5}) + 2\delta < 1$, then Propositions II. 2.2 (with $N = 1$ and $\theta = 2/\sqrt{5}$) and II. 2.5 may be used to imply that the operator V_{T_α,ρ_β} has simple, singular spectrum.

Thus, to complete the proof, it only remains to note, as a particular consequence of Proposition 1.3, that, for any $\delta > 0$,

$\{\beta \in [0,1) : \langle\langle q_n \beta\rangle\rangle \leq \delta$ for infinitely many $n\}$ is a residual, measure-one subset of $[0,1)$.

Remark 2.2.

It is clear from the above proof that, given any function ρ which is constant on both $[0,\beta)$ and $[\beta,1)$, then the operator $V_{T_\alpha,\rho}$ will have simple, singular spectrum if

$$\liminf_{n\to\infty} \langle\langle q_n \beta\rangle\rangle < \tfrac{1}{2}(1 - 2/\sqrt{5}).$$

Now, adopting the notation of II. 4, let $\Lambda(= \{e^{2\pi in\alpha} : n \in Z\})$ be the group of eigenvalues of T_α. The proof of Proposition II. 4.1 (b) shows how to construct a function $\rho : [0,1) \to Z_2$ which is cohomologous to any preassigned $\lambda \in \Lambda^{\frac{1}{2}} = \{\pm e^{\pi in\alpha} : n \in Z\}$. As a particular case of this construction, it follows that the function $\rho_\alpha \circ T_\alpha$ is cohomologous to $\lambda = e^{\pi i\alpha}$. From this it is clear that ρ_α itself must be cohomologous to the constant $e^{\pi i\alpha}$. On the other hand, for any even integer n, it may be shown that $\rho_{<n\alpha>}$ is cohomologous to $(-1)^{[n\alpha]}$. (Consider $\psi \circ T_\alpha/\psi$, where

$$\psi = \rho_\alpha \circ T_\alpha^{-1} \cdot \rho_\alpha \circ T_\alpha^{-3} \quad \cdots \quad \cdot \rho_\alpha \circ T^{-n+1}.$$

This works for $n > 0$. Similar reasoning applies for $n < 0$.) Multiplying together in a suitable way, it follows that if n is any integer and $\beta = n\alpha \pmod 1$, then ρ_β is cohomologous to one of the four constants $e^{\pi i\alpha}$, $-e^{\pi i\alpha}$, 1, -1, or, equivalently, that V_{T_α, ρ_β} has purely discrete spectrum (disjoint from that of T_α, unless n and $[n\alpha]$ are both even).

In his approach to the question of the ergodicity of the skew products $(T_\alpha)_{\rho_\beta} : [0,1) \times Z_2 \to [0,1) \times Z_2$ (see Definition II. 1.1), Veech [24] has made a detailed study of the question of when (i.e. for what values of the parameters) the operator V_{T_α, ρ_β} satisfies condition (b) of Proposition II. 4.4, viz. the condition that

$$\lim_{\langle\langle n\alpha \rangle\rangle \to 0} \left| \left\langle V^n_{T_\alpha, \rho_\beta} 1, \quad 1 \right\rangle \right| = 1.$$

Using II. 4.4, Veech's results imply, in particular, that

(i) if α has bounded partial quotients in its continued fraction expansion, then V_{T_α,ρ_β} has discrete spectrum if and only if

$$\beta = n\alpha \pmod 1 \qquad \text{for some } n \in \mathbb{Z},$$

and

(ii) if α has unbounded partial quotients in its continued fraction expansion, then those $\beta \in [0,1)$ for which V_{T_α,ρ_β} has discrete spectrum form an uncountable, measure-zero subgroup of $[0,1)$.

Thus, for any given irrational $\alpha \in (0,1)$, the operator V_{T_α,ρ_β} has purely continuous spectrum for almost all values of β. It is not difficult to show that, in addition, these values of β always form a residual subset of $[0,1)$.

The following corollary to Proposition 2.1 is now an easy consequence of the decomposition

$$U_{(T_\alpha)_{\rho_\beta}} \cong U_{T_\alpha} \oplus V_{T_\alpha,\rho_\beta} .$$

PROPOSITION 2.3.

For any given irrational $\alpha \in (0,1)$, the skew product transformation $(T_\alpha)_{\rho_\beta} : [0,1) \times Z_2 \to [0,1) \times Z_2$ has simple, singular spectrum for all values of β in some residual, measure-one subset of $[0,1)$.

Proof

Use 2.1 and the almost-everywhere, residual disjointness of the spectra of U_{T_α} and V_{T_α,ρ_β}.

Remark 2.4.

By a different approach, Katok and Stepin [11,12] obtained the results of 2.1 and 2.3 under certain more restrictive conditions on α and β. In [11], it is stated that if α and β are rationally independent, then the spectral multiplicity of $(T_\alpha)_{\rho_\beta}$ does not exceed two. See also Oseledec [18] for the result that, in the special case when β is $\frac{1}{2}$, the operator V_{T_α, ρ_β} has continuous spectrum with spectral multiplicity not exceeding two, for all irrational α.

§3. A Class of Cylinder Transformations - (a) Spectral Properties

Consider the class of measure-preserving cylinder automorphisms $\{T_{\alpha,\beta}: \alpha, \beta \in (0,1)\}$, where each $T_{\alpha,\beta}$ is defined to be the skew product $\mathbb{R}$-extension of the rotation T_α determined by $\chi_{[0,\beta)} - \beta$, i.e. (see Definition II. 1.1) where

$$T_{\alpha,\beta}(x,t) = (x+\alpha \pmod 1), \quad t + \chi_{[0,\beta)}(x) - \beta)$$

$$\text{for all } (x,t) \in [0,1) \times \mathbb{R}.$$

The space $[0,1) \times \mathbb{R}$ is assumed to be equipped with the product, denoted μ , of the Lebesgue measures on $[0,1)$ and $\mathbb{R}$ respectively.

If ℓ is a positive integer, then $T_{\alpha,\beta}^{\ell}$ acts on a point (x,t) in $[0,1) \times \mathbb{R}$ as follows:

$$T_{\alpha,\beta}^{\ell}(x,t) = (\langle x + \ell\alpha \rangle, \quad t + \sum_{i=0}^{\ell-1} \chi_{[0,\beta)}(\langle x + i\alpha \rangle)$$

$$- \ell\beta).$$

In order to estimate the vertical components of the translations brought about by iterates of $T_{\alpha,\beta}$, we introduce the sequence of "discrepancies"

$$D_\ell(\alpha) = \sup_{0 \le a < b \le 1} \left| (1/\ell) \sum_{i=0}^{\ell-1} \chi_{[a,b)}(\langle i\alpha \rangle) - (b-a) \right| ,$$

$$= 1,2,\ldots \quad .$$

Note that, for each positive integer ℓ, the vertical distance through which $T_{\alpha,\beta}^{\ell}$ shifts a point in $[0,1) \times \mathbb{R}$ is never more than $2\ell D_\ell(\alpha)$. The sequences of discrepancies, $D_\ell(\alpha)$, $\ell = 1,2,\ldots$, α irrational, have long been studied in connection with the irregularity of distribution of the sequences $i\alpha$, $i = 1,\ldots$ (see Kuipers and Niederreiter [16]). We shall have use of the well known result that, for any irrational α, the sequence $\ell D_\ell(\alpha)$, $\ell = 1,2,\ldots$, is unbounded.

PROPOSITION 3.1.

Suppose that α is an element of $(0,1)$ for which there exists a sequence of irreducible fractions p_n/q_n, $n = 1,2,3,\ldots$, such that, as $n \to \infty$,

(i) $q_n \nearrow \infty$

(ii) $s_n q_n^2 | \alpha - p_n/q_n | \to 0$, where

$$s_n = \sup_{1 \le \ell \le q_n} \ell D_\ell(\alpha) \text{ for each } n.$$

Then, whenever $\beta \in (0,1)$ satisfies

$$\text{(iii)} \liminf_{n\to\infty} \max\left(\langle\!\langle q_n\beta \rangle\!\rangle , \ (1/\langle\!\langle q_n\beta \rangle\!\rangle).s_n q_n^2 | \alpha - p_n/q_n | \right) = 0,$$

the transformation $T_{\alpha,\beta}$ has spectral multiplicity uniformly equal to one.

Proof

Note that in order that condition (iii) be satisfied, β must be irrational. Hence $\langle\!\langle q_n\beta \rangle\!\rangle$ is non-zero for all n. Also, by going to a subsequence of the given sequence of irreducible fractions, we may assume that β satisfies

$$\text{(iii)'} \begin{cases} \lim\limits_{n\to\infty} \langle\!\langle q_n\beta \rangle\!\rangle = 0 \ , \\[2ex] \lim\limits_{n\to\infty} (s_n/\langle\!\langle q_n\beta\rangle\!\rangle) . \ q_n^2 \ | \alpha - p_n/q_n | = 0. \end{cases}$$

We will show that, under these conditions, the transformation $T_{\alpha,\beta}$ admits a multiplicity one approximation with speed $o(1/n)$ (see Definition I. 2.5').

Fix a positive integer n, and split the space $[0,1) \times \mathbb{R}$ into a disjoint union of the $2q_n$ "columns" of the two "types" $E_k(n)$ and $F_k(n)$, $k = 0,\ldots,q_n-1$, defined by

$$E_k(n) = [k/q_n, \ \frac{k + \langle q_n\beta\rangle}{q_n}) \times \mathbb{R}$$

and

$$F_k(n) = [\frac{k + \langle q_n\beta\rangle}{q_n}, \ \frac{k+1}{q_n}) \times \mathbb{R}.$$

The effect of the transformation $T_{p_n/q_n,\beta}$ on any of these columns is a rigid translation with horizontal component p_n/q_n (mod 1) and vertical component either $1 - \beta$ or $-\beta$, according to whether the column in question lies to the left or right, respectively, of the vertical line

$$L = \{(x,t) \in [0,1) \times \mathbb{R} : x = \beta\}.$$

Note that each column lies entirely to one side or other of L. Focussing on the horizontal component of this translation, we see that for each $k \in \{0,\ldots,q_n-1\}$,

$$T_{p_n/q_n,\beta}\, E_k(n) = E_{k+p_n(\text{mod } q_n)}(n)$$

and

$$T_{p_n/q_n,\beta}\, F_k(n) = F_{k+p_n(\text{mod } q_n)}(n).$$

Thus, the irreducibility of the fraction p_n/q_n implies that, as a permutation of either $\{E_o(n),\ldots,E_{q_n-1}(n)\}$ or $\{F_o(n),\ldots,F_{q_n-1}(n)\}$, $T_{p_n/q_n,\beta}$ is cyclic. Hence, to determine the net translation of a column under q_n iterations of $T_{p_n/q_n,\beta}$, it is necessary only to count the number of columns of the same type lying on either side of L. The columns to the left of L are $E_o(n),\ldots,E_{[q_n\beta]}$ and $F_o(n),\ldots,F_{[q_n\beta]-1}(n)$. Upon making the calculations

$$([q_n\beta] + 1)(1-\beta) - (q_n-([q_n\beta] + 1))\beta = 1 - \langle q_n\beta\rangle$$

and

$$[q_n\beta](1-\beta) - (q_n - [q_n\beta])\beta = -\langle q_n\beta\rangle,$$

we conclude that $T^{q_n}_{p_n/q_n,\beta}$ translates each $E_k(n)$, $k = 0,\ldots,q_n-1$, vertically upwards by $1 - \langle q_n\beta\rangle$ units, whereas each $F_k(n)$, $k = 0,\ldots,q_n-1$ is translated vertically downwards by $\langle q_n\beta\rangle$ units.

The above observations allow us to use $T_{p_n/q_n,\beta}$ to define our n-th approximating semi-partition,

$$\xi(n) = \{C_i(n) : i = 1,\ldots,6r_nq_n\},$$

by choosing

$$C_i(n) = T^{i-1}_{p_n/q_n,\beta}\; C_1(n), \text{ for each } i \in \{1,\ldots,6r_nq_n\},$$

with $r_n = [(s_n+1)/ \langle\langle q_n\beta\rangle\rangle]$

and

$$C_1(n) = \begin{cases} \{(x,t) \in E_0(n): -3r_n \langle\langle q_n\beta\rangle\rangle \le t < (-3r_n+1) \langle\langle q_n\beta\rangle\rangle\}, \\ \qquad \text{if } \langle\langle q_n\beta\rangle\rangle = 1 - \langle q_n\beta\rangle, \\ \{(x,t) \in F_0(n): (3r_n-1) \langle\langle q_n\beta\rangle\rangle \le t < 3r_n \langle\langle q_n\beta\rangle\rangle\}, \\ \qquad \text{if } \langle\langle q_n\beta\rangle\rangle = \langle q_n\beta\rangle. \end{cases}$$

Each element of $\xi(n)$ is a rectangle of base $(1 - \langle\langle q_n\beta\rangle\rangle)/q_n$, filling a horizontal strip of depth $\langle\langle q_n\beta\rangle\rangle$ across the column in which it lies. In the case when $\langle\langle q_n\beta\rangle\rangle = 1 - \langle q_n\beta\rangle$ (respectively $\langle q_n\beta\rangle$), the rectangles $C_1(n),\dots,C_{q_n}(n)$ lie, one in each of the columns $E_0(n),\dots,E_{q_n-1}(n)$ (respectively $F_0(n),\dots,F_{q_n-1}(n)$), forming a pattern in which no element is vertically displaced with respect to $C_1(n)$ by more than $2s_n$-units (see the remarks accompanying the definition of the sequence of discrepancies). The other elements of $\xi(n)$ may be obtained by taking $6r_n-1$ successive vertical shifts of this basic pattern, upward (respectively downward) by $\langle\langle q_n\beta\rangle\rangle$ units. This implies that the elements of $\xi(n)$ are indeed disjoint, and by the choice of r_n, that

$$(*) \quad \bigcup_{C\in\xi(n)} C \supset \begin{cases} \left(\bigcup_{k=0}^{q_n-1} E_k(n)\right) \cap ([0,1) \times [-s_n,s_n)), \\ \qquad \text{if } \langle\langle q_n\beta\rangle\rangle = 1 - \langle q_n\beta\rangle, \\ \left(\bigcup_{k=0}^{q_n-1} F_k(n)\right) \cap ([0,1) \times [-s_n,s_n)), \\ \qquad \text{if } \langle\langle q_n\beta\rangle\rangle = \langle q_n\beta\rangle \;. \end{cases}$$

Note that, for all Borel subsets A of $[0,1) \times \mathbb{R}$ with $\mu(A) < \infty$, the following hold:

$$\lim_{\langle q_n\beta\rangle \to 1} \mu\Big(\Big(\bigcup_{k=0}^{q_n-1} E_k(n)\Big) \cap A\Big) = \mu(A)$$

and

$$\lim_{\langle q_n\beta\rangle \to 0} \mu\Big(\Big(\bigcup_{k=0}^{q_n-1} F_k(n)\Big) \cap A\Big) = \mu(A).$$

Since $s_n \to \infty$, and the dimensions of the rectangles in $\xi(n)$ tend uniformly to zero as $n \to \infty$, we can thus conclude from (*) that $\xi(n) \to \varepsilon$ as $n \to \infty$.

Furthermore,

$$(1/6r_nq_n\ \mu(C_1(n))).\ \sum_{i=1}^{6r_nq_n-1} \mu(T_{\alpha,\beta}\ C_i(n)\ \Delta\ C_{i+1}(n))$$

$$= \frac{(6r_nq_n-1).2\ |\alpha-p_n/q_n|\ \langle\langle q_n\beta\rangle\rangle}{6r_nq_n\ .((1-\langle\langle q_n\beta\rangle\rangle)/q_n)\ \langle\langle q_n\beta\rangle\rangle}$$

$$= (1/6r_nq_n).\ \frac{(12r_nq_n-2)q_n|\alpha-p_n/q_n|}{(1-\langle\langle q_n\beta\rangle\rangle)}$$

$$= o(1/6r_nq_n)\ ,$$ by (iii)' and the choice of r_n, $n = 1,2,\ldots$.

Hence, the semi-partitions $\xi(n)$, $n = 1,2,\ldots$, define a multiplicity one approximation of $T_{\alpha,\beta}$ with speed $o(1/n)$. It only remains to apply Theorem I. 3.1 (or II. 2.2).

Under the conditions of Proposition 3.1, the transformation $T_{\alpha,\beta}$ must have singular spectral type. This follows from:

PROPOSITION 3.2.

Suppose that α and β are irrational elements of $(0,1)$ for which there exists a sequence of irreducible fractions p_n/q_n, $n = 1,2,\ldots$, with

(i) $q_n \nearrow \infty$, as $n \to \infty$,

(ii) for all n, $2q_n^2 |\alpha - p_n/q_n| \leq \theta < 1$, where θ is some constant, independent of n,

(iii) $\liminf_{n \to \infty} \langle\langle q_n\beta \rangle\rangle = 0$.

Then the cylinder transformation $T_{\alpha,\beta}$ has singular spectral type.

Proof

Choose a positive constant, a, with $\theta + 2/a < 1$. By going to a subsequence of the given sequence of irreducible fractions, we may assume that

$$\langle\langle q_n\beta \rangle\rangle \to 0 \quad \text{as } n \to \infty,$$

and that there exists a constant $\theta' < 1$ with

$$\theta + (2/a) + 2\langle\langle q_n\beta \rangle\rangle \leq \theta', \text{ for each } n. \tag{1}$$

Consider an arbitrary rectangle in $[0,1) \times \mathbb{R}$ of the form

$$C = [k/q_n, (k+1)/q_n) \times [t, t+a\langle\langle q_n\beta \rangle\rangle),$$

$$\text{with } k \in \{0,\ldots,q_n-1\}, \quad t \in \mathbb{R}.$$

Let C' be the subrectangle defined by:

$$C' = \begin{cases} [k/q_n, (k+\langle q_n\beta\rangle)/q_n) \times [t, t+a\langle\!\langle q_n\beta\rangle\!\rangle), \\ \qquad\qquad \text{if } \langle\!\langle q_n\beta\rangle\!\rangle = 1 - \langle q_n\beta\rangle, \\ [(k+\langle\!\langle q_n\beta\rangle\!\rangle)/q_n, (k+1)/q_n) \times [t, t+a\langle\!\langle q_n\beta\rangle\!\rangle), \\ \qquad\qquad \text{if } \langle\!\langle q_n\beta\rangle\!\rangle = \langle q_n\beta\rangle. \end{cases}$$

Note that

$$\mu(C \,\Delta\, C') = \langle\!\langle q_n\beta\rangle\!\rangle\, \mu(C). \tag{2}$$

From the proof of Proposition 3.1, it is clear that $T^{q_n}_{p_n/q_n,\beta}\, C'$ is just a vertical translate of C' by $\pm\ \langle\!\langle q_n\beta\rangle\!\rangle$ units.

Hence,

$$\mu(T^{q_n}_{p_n/q_n,\beta}\, C' \,\Delta\, C') = (2/a)\, \mu(C'). \tag{3}$$

Now,

$$\mu(T^{q_n}_{\alpha,\beta}\, C' \,\Delta\, T^{q_n}_{p_n/q_n,\beta}\, C')$$

$$\leq \sum_{i=0}^{q_n-1} \mu(T^{q_n-i}_{\alpha,\beta}\ T^{i}_{p_n/q_n,\beta}\, C' \,\Delta\, T^{q_n-i-1}_{\alpha,\beta}\ T^{i+1}_{p_n/q_n,\beta}\, C')$$

$$= \sum_{i=0}^{q_n-1} \mu(T_{\alpha,\beta}\cdot T^{i}_{p_n/q_n,\beta} C' \,\Delta\, T_{p_n/q_n,\beta}\cdot T^{i}_{p_n/q_n,\beta}\, C')$$

$$= q_n \cdot 2|\alpha - p_n/q_n|\ \ a\ \langle\!\langle q_n\beta\rangle\!\rangle$$

$$\leq \theta\mu(C)\ , \qquad \text{by condition (ii).} \tag{4}$$

Here, we have used the fact that under any iterate of $T_{p_n/q_n,\beta}$, the rectangle C' is translated to another rectangle of the same type. This is apparent from the proof of Proposition 3.1.

Together, (1), (2), (3) and (4) imply that, for any rectangle C of the assumed type,

$$\mu(T_{\alpha,\beta}^{q_n} C \,\Delta\, C) \leq \theta' \mu(C).$$

Since β is irrational, $\langle\langle q_n\beta\rangle\rangle$ is never zero, and the proof of the proposition may be completed by applying Theorem I. 3.4. to the sequence of partitions defined as follows:

$$\xi(n) = \{[k/q_n, (k+1)/q_n) \times [\ell a \langle\langle q_n\beta\rangle\rangle, (\ell+1)a \langle\langle q_n\beta\rangle\rangle)$$

$$: k = 0,\ldots,q_n-1, \quad \ell \in Z\}, \text{ for all } n = 1,2,\ldots .$$

Now, consider the case when the parameter β is rational, say equal to c/d in lowest terms. Then, for all α in (0,1), any vertical translate of $[0,1) \times d^{-1}.Z$ is a $T_{\alpha,\beta}$ -invariant subset of $[0,1) \times \mathbb{R}$. Thus, we are lead to define a new measure-preserving automorphism, denoted $S_{\alpha,c/d}$, by taking the restriction of $T_{\alpha,c/d}$ to the space $[0,1) \times d^{-1}.Z$ (equipped with the obvious product of Lebesgue and counting measures). The class of transformations $\{S_{\alpha,c/d} : \alpha, c/d \in (0,1)\}$ is of interest in that it provides examples of approximations of all finite multiplicities:

PROPOSITION 3.3.

Let c/d be an irreducible fraction in (0,1). Suppose that $\alpha \in (0,1)$ is such that there exists a sequence of irreducible fractions p_n/q_n, $n = 1,2,\ldots$, satisfying

(i) no q_n is a multiple of d,

(ii) $q_n \nearrow \infty$ as $n \to \infty$,

and

(iii) $s_n q_n^2 |\alpha - p_n/q_n| \to 0$, as $n \to \infty$, where, as before,

$$s_n = \sup_{1 \le \ell \le q_n} \ell D_\ell(\alpha), \text{ for each } n.$$

Then the transformation $S_{\alpha,c/d}$ admits a multiplicity d approximation with speed $o(1/n)$ and, hence, has spectral multiplicity less than or equal to d.

Proof

Put $b_n = d\langle q_n c/d\rangle$. Then, by (i), b_n belongs to $\{1,2,\ldots,d-1\}$, for each $n = 1,2,\ldots$.

From the proof of Proposition 3.1 it is apparent that the columns, defined this time by

$$E_k(n) = [k/q_n, k/q_n + b_n/dq_n) \times d^{-1}.Z$$

and

$$F_k(n) = [k/q_n + b_n/dq_n, (k+1)/q_n) \times d^{-1}.Z,$$

for $k \in \{0,\ldots,q_n-1\}$, have the following properties with respect to the restricted transformation $S_{p_n/q_n,c/d}$:

(a) each column is rigidly translated by $S_{p_n/q_n,c/d}$ onto another column of the same type,

(b) as a permutation of either $\{E_o(n),\ldots,E_{q_n-1}(n)\}$ or $\{F_o(n),\ldots,F_{q_n-1}(n)\}$, $S_{p_n/q_n,c/d}$ is cyclic,

(c) under $S^{q_n}_{p_n/q_n,c/d}$, each of the columns $E_o(n),\ldots,E_{q_n-1}(n)$ moves vertically upwards by $1 - \langle q_n c/d\rangle$ units ($d-b_n$ "levels"), and

(d) under $S^{q_n}_{p_n/q_n,c/d}$, each of the columns $F_o(n),\ldots,F_{q_n-1}(n)$ is translated vertically downwards by $\langle q_n c/d\rangle$ units (b_n "levels").

Note that by a level of a column we mean one of the doubly infinite stack of intervals of which it is composed.

These properties lead us to choose our sequence of approximating semi-partitions as follows:

For each $n = 1,2,\ldots,$ we put

$$\xi(n) = \{C_{ij}(n) : i = 1,\ldots,6r_nq_n;\ j = 1,\ldots d\},$$

where we define

$$C_{ij}(n) = S^{i-1}_{p_n/q_n,c/d}\, C_{1j}(n), \text{ for all } i \text{ and } j,$$

with

$$r_n = d[s_n + 1]$$

and

$$C_{1j}(n) = \begin{cases} [0,b_n/dq_n) \times \{(-3r_n+j)/d\}, & \text{for } j = 1,\ldots, d-b_n, \\ [b_n/dq_n\ ,\ 1/q_n) \times \{(3r_n+j-d)/d\}, & \text{for } j = d-b_n+1,\ldots,d. \end{cases}$$

The intervals $C_{i1}(n),\dots,C_{i,d-b_n}(n)$ (respectively $C_{1,d-b_n+1}(n),\dots,C_{1d}(n)$) have been chosen to be successive levels of the column $E_o(n)$ (respectively $F_o(n)$). This implies, using (a) and (b) above, that $\xi(n)$ admits the following subdivision by columns:

For each $k \in \{1,\dots,q_n\}$, whenever $i \in \{k + \ell q_n: \ell = 0,\dots,6r_n-1\}$, the elements $C_{i,1}(n),\dots,C_{i,d-b_n}(n)$ (respectively $C_{i,d-b_n+1}(n),\dots,C_{i,d}(n)$) are successive levels of the column $S^{k-1}_{p_n/q_n,c/d}\, E_o(n)$ (respectively $S^{k-1}_{p_n/q_n,c/d}\, F_o(n)$). Note that each of the columns $E_o(n),\dots,E_{q_n-1}(n)$ (respectively $F_o(n),\dots,F_{q_n-1}(n)$) is of the form $S^{k-1}_{p_n/q_n,c/d}\, E_o(n)$ (respectively $S^{k-1}_{p_n/q_n,c/d}\, F_o(n)$), for some $k \in \{1,\dots,q_n\}$.

Now, for fixed $k \in \{1,\dots,q_n\}$, property (c) (respectively (d)), above, implies that if $i \in \{k+\ell q_n : \ell = 0,\dots,6r_n-2\}$, then the intervals $C_{i+q_n,1}(n), C_{i+q_n,2}(n),\dots,C_{i+q_n,d-b_n}(n)$ (respectively $C_{i+q_n,d-b_n+1}(n),\dots,C_{i+q_n,d}(n)$) occupy the next $(d-b_n)$-tuple of successive levels of $S^{k-1}_{p_n/q_n,c/d}\, E_o(n)$ above (respectively the next b_n-tuple of successive levels of $S^{k-1}_{p_n/q_n,c/d}\, F_o(n)$ below) that occupied by $C_{i,1}(n),\dots,C_{i,d-b_n}(n)$ (respectively $C_{i,d-b_n+1}(n),\dots,C_{i,d}(n)$). Thus, within the columns $E_o(n),\dots,E_{q_n-1}(n)$ (respectively

$F_o(n),\ldots,F_{q_n-1}(n))$, the semi-partition $\xi(n)$ contains stacks of $6r_n(d-b_n)$ (respectively $6r_nb_n$) consecutive levels. The lowermost (respectively uppermost) levels in these stacks are, modulo a reordering, $C_{1,1}(n)$, $C_{2,1}(n),\ldots,C_{q_n,1}(n)$ (respectively $C_{1,d}(n)$, $C_{2,d}(n),\ldots,C_{q_n,d}(n))$, none of which, by the definition of s_n, is more than $2s_n$ units vertically displaced from the $3r_n{}^{th}$ level below (respectively above) the horizontal axis. Since, by the definition of r_n, a vertical displacement of $2s_n$ units spans no more than $2r_n$ levels, we see that

$$\{(x,j/d) : x \in [0,1),\ j \in \{-r_n,-r_n+1,\ldots,r_n\}\}$$

$$\subset \bigcup_{C\in\xi(n)} C, \quad \text{for all } n.$$

Together with the uniform convergence to zero of the lengths of the elements of $\xi(n)$, this inclusion implies that $\xi(n) \to \varepsilon$ as $n \to \infty$.

The rest of the proof is exactly as for 3.1. The details are left to the reader.

The following proposition may be proved in the same way as Proposition 3.2.

<u>PROPOSITION 3.4.</u>

Let α be an element of $(0,1)$ for which there exists a sequence of irreducible fractions p_n/q_n, $n = 1,2,\ldots$, satisfying

(i) each q_n is a multiple of d,

(ii) $q_n \nearrow \infty$, as $n \to \infty$,

(iii) there exists a constant $\theta < 1$ with

$$2q_n^{\,2}\, | \alpha - p_n/q_n | \le \theta, \text{ for all } n.$$

Then the transformation $S_{\alpha,c/d}$ has singular spectral type.

Remark 3.5.

The conditions on the parameter α hypothesized in Propositions 3.1 - 3.4, respectively, are satisfied on residual subsets of the parameter space (0,1). In the cases of 3.2 and 3.4, Proposition 1.2 implies that these subsets have Lebesgue measure one.

It follows from Proposition 1.3, that, for any α which satisfies the conditions of either 3.1 or 3.2, the corresponding condition on the parameter β is satisfied by a residual, measure-one subset of values in (0,1).

Remark 3.6.

From the proofs of Propositions 3.1 and 3.3, the transformations $T_{p_n/q_n,\beta}$ and $S_{p_n/q_n,c/d}$ used in approximating $T_{\alpha,\beta}$ and $S_{\alpha,c/d}$, respectively, may be seen to have infinite Lebesgue spectrum (in fact, these transformations are dissipative - each having a wandering set whose transforms cover the whole space). The question arises whether either of the transformations $T_{\alpha,\beta}$ or $S_{\alpha,c/d}$ may have Lebesgue spectral type (with finite multiplicity?) for some irrational value of α ?

The following proposition summarizes a number of spectral observations which do not depend upon the method of approximations. Note that the maximal spectral type and spectral multiplicity functions of any measure-preserving transformation are defined on K, the circle group.

PROPOSITION 3.7.

Let α and β be arbitrary elements of (0,1). Then

(a) the transformation $T_{\alpha,\beta}$ has maximal spectral type and spectral multiplicity function both invariant under each of the following transformations of the circle group:

(i) $z \mapsto \overline{z}$, $z \in K$, the reflection in the horizontal axis,

(ii) $z \mapsto e^{2\pi i\alpha}.z$, $z \in K$, the rotation through the angle $2\pi\alpha$,

(iii) $z \mapsto e^{2\pi i\beta}.z$, $z \in K$, the rotation through the angle $2\pi\beta$;

(b) when β is rational, the same is true of the maximal spectral type and spectral multiplicity function of the transformation $S_{\alpha,\beta}$;

(c) in the special case when β equals $\frac{1}{2}$, the spectral multiplicity function of $S_{\alpha,\beta}$ is even almost everywhere on K.

Proof

For each $t \in \mathbb{R}$, let $V_{\alpha,\beta,t}$ denote the unitary operator defined as follows on $L^2[0,1)$:

$$(V_{\alpha,\beta,t}y)(x) = \exp(-2\pi it\,(\chi_{[0,\beta)}(x)-\beta)).y(T_\alpha x),$$

for all $y \in L^2[0,1)$ and $x \in [0,1)$.

Then Proposition II. 1.6 implies that the unitary operator induced on $L^2([0,1) \times \mathbb{R})$ by $T_{\alpha,\beta}$ is isomorphic to the direct integral $\int_{\mathbb{R}}^{\oplus} V_{\alpha,\beta,t}\,dt$; and that, if c/d is an irreducible fraction in (0,1), then the unitary operator induced on $L^2([0,1) \times d^{-1}.Z)$ by $S_{\alpha,c/d}$ is isomorphic to $\int_{[0,d)}^{\oplus} V_{\alpha,c/d,t}\,dt$.

Now, check that the identities

(i) $W_1^* V_{\alpha,\beta,t} W_1 = V^*_{\alpha,\beta,-t}$ $(= V^*_{\alpha,c/d,d-t},$ if $\beta = c/d)$,

(ii) $W_2^* V_{\alpha,\beta,t} W_2 = e^{2\pi i\alpha} V_{\alpha,\beta,t}$, and

(iii) $V_{\alpha,\beta,t-1} = e^{2\pi i\beta} V_{\alpha,\beta,t}$

hold for all possible values of the parameters, where W_1 and W_2 are the unitary operators on $L^2[0,1)$ defined by setting

$$W_1y(x) = y(\alpha+\beta-x \pmod 1)$$

and

$$W_2y(x) = e^{2\pi ix}\,y(x), \text{ for all } y \in L^2[0,1) \text{ and } x \in [0,1).$$

The identities (i), (ii) and (iii), applied to the stated direct integral decompositions of the unitary operators induced from $T_{\alpha,\beta}$ and $S_{\alpha,\beta}$, imply that each of these operators is unitarily equivalent to (i) it's adjoint, (ii) itself multiplied by $e^{2\pi i\alpha}$, and (iii) itself multiplied by $e^{2\pi i\beta}$, respectively. These unitary equivalences prove parts (a) and (b) of the statement of the proposition.

Now, note that

$$W_3^* V_{\alpha,\frac{1}{2},t} W_3 = V_{\alpha,\frac{1}{2},2-t}, \text{ for all } \alpha \in (0,1),\ t \in \mathbb{R},$$

where W_3 is the unitary operator defined on $L^2[0,1)$ as follows:

$$W_3 y(x) = y(x + \tfrac{1}{2} \pmod 1), \text{ for all } y \in L^2[0,1) \text{ and } x \in [0,1).$$

It follows from this identity that the direct integral decomposition of the unitary operator induced from $S_{\alpha,\frac{1}{2}}$ splits as the direct sum of two isomorphic parts – $\int_{[0,1)}^{\oplus} V_{\alpha,\frac{1}{2},t}\,dt$ and $\int_{[1,2)}^{\oplus} V_{\alpha,\frac{1}{2},t}\,dt$. This proves (c).

COROLLARY 3.8.

If α satisfies the conditions of Proposition 3.3 with $c/d = \frac{1}{2}$, then the spectral multiplicity of the transformation $S_{\alpha,\frac{1}{2}}$ is uniformly equal to two.

Problem 3.9.

Is it possible to conclude, in general, that under the conditions of Proposition 3.3, the spectral multiplicity function of $S_{\alpha,c/d}$ is uniformly equal to d?

Remark 3.10.

From the direct integral decomposition used in the proof of 3.7, one may deduce that if $T_{\alpha,\beta}$ has simple singular spectrum, in particular if α and β satisfy the conditions of Proposition 3.1, then the following holds: there exists a null subset N of $\mathbb{R}$ such that for each $t \in \mathbb{R}\backslash N$, the operator $V_{\alpha,\beta,t}$ has simple, singular spectrum, disjoint from that of any other $V_{\alpha,\beta,t'}$ with $t' \in \mathbb{R}\backslash N$.

Remark 3.11.

That part of 3.7 which asserts the invariance under the rotation by $2\pi\alpha$ is a special case of the following general observation:

If T is a measure-preserving automorphism of a finite measure space, then the spectral properties of any skew product extension of T, or of any operator of the form $V_{T,\rho}$, are always invariant under the group of rotations of K defined by the eigenvalues of T.

§4. A Class of Cylinder Transformations - (b) Ergodicity

The results of this section will be submitted for publication as a joint paper - with Giles Atkinson as co-author. Proposition 4.4 and a weaker form of Proposition 4.1 have already appeared in [1] - with different proofs. Lemma 4.2, the key to our present approach, is entirely the work of the current author.

For each α and β in $(0,1)$, let $\phi_\beta = \chi_{[0,\beta)} - \beta$; and let $T_{\alpha,\beta} = (T_\alpha)_{\phi_\beta}$ be as defined in the previous section. We shall prove:

PROPOSITION 4.1.

For any given irrational $\alpha \in (0,1)$, the cylinder transformation $T_{\alpha,\beta}: [0,1) \times \mathbb{R} \to [0,1) \times \mathbb{R}$ is ergodic for all values of β in some residual, measure-one subset of $(0,1)$.

For the proof of this proposition the value of the irrational parameter α shall be supposed fixed. The main step in this proof is contained in:

LEMMA 4.2.

Suppose that p_n/q_n, $n = 1,2,\ldots$ is a sequence of irreducible fractions such that $q_n \nearrow \infty$ as $n \to \infty$, and

$$q_n^2 | \alpha - p_n/q_n | \leq \theta, \text{ for all } n, \tag{1}$$

where θ is a positive constant which is strictly less than one. If β is such that the sequence $\langle q_n\beta \rangle$, $n = 1,2,\ldots$ has a limit point $t \in (0,1)$, then:

(a) $E_{T_\alpha}(\phi_\beta)$ contains t whenever $0 < t < 1-\theta$;

(b) $E_{T_\alpha}(\phi_\beta)$ contains $1-t$ whenever $\theta < t < 1$.

(See II. 3 for the definition of the set of essential values $E_{T_\alpha}(\phi_\beta)$; and its importance to the ergodicity of $T_{\alpha,\beta}$.)

Proof

By considering a suitable subsequence of the fractions p_n/q_n, we may assume that

$$\lim_{n\to\infty} \langle q_n\beta\rangle = t . \tag{2}$$

Consider the following sequences of partitions of $[0,1)$:

$$\xi(n) = \{[i/q_n), [(i+1)/q_n) : i = 0,\ldots,q_n-1\} \text{ for } n=1,2,\ldots .$$

Let C be an element of $\xi(n)$. Then the set $C \cap T_\alpha^{-q_n} C$ is a sub-interval of C with

$$T_\alpha^j (C \cap T_\alpha^{-q_n} C) \subset T_{p_n/q_n}^j C, \tag{3}$$

for all $j \in \{0,1,\ldots,q_{n-1}\}$. Also, for these j,

$$\mu(T_{p_n/q_n}^j C \setminus T_\alpha^j (C \cap T_\alpha^{-q_n} C)) \le \theta.\mu(C). \tag{4}$$

Since p_n/q_n is irreducible, T_{p_n/q_n} cyclically permutes the elements of $\xi(n)$. So, corresponding to each $i \in \{0,1,\ldots,q_n-1\}$ there is a unique $j(i) \in \{0,\ldots,q_n-1\}$ with

$$T_{p_n/q_n}^{j(i)} C = [i/q_n, (i+1)/q_n).$$

From (3) and the inequality $[q_n\beta]/q_n \le \beta < ([q_n\beta] + 1)/q_n$,

it follows that

$$T_\alpha^{j(i)}(C \cap T_\alpha^{-q_n}C) \subset \begin{cases} [0,\beta), \text{ for } i = 0,\ldots,[q_n\beta] - 1, \\ [\beta,1), \text{ for } i = [q_n\beta] + 1,\ldots,q_n-1. \end{cases}$$

So, for all $x \in C \quad T_\alpha^{-q_n}C$,

$$\sum_{j=0}^{q_n-1} (\chi_{[0,\beta)} - \beta)(T_\alpha^j x)$$

$$= \left(\sum_{i=0}^{q_n-1} \chi_{[0,\beta)}(T_\alpha^{j(i)}x)\right) - q_n\beta$$

$$= [q_n\beta] + \chi_{[0,\beta)}(T_\alpha^{j([q_n\beta])}x) - q_n\beta$$

Hence,

$$\sum_{j=0}^{q_n-1} \phi_\beta(T_\alpha^j x)$$

$$= \begin{cases} 1 - \langle q_n\beta\rangle, & \text{for } x \in C \cap T_\alpha^{-q_n}C \cap T_\alpha^{-j([q_n\beta])}[0,\beta), \\ -\langle q_n\beta\rangle, & \text{for } x \in C \cap T_\alpha^{-q_n}C \cap T_\alpha^{-j([q_n\beta])}[\beta,1). \end{cases} \tag{5}$$

Now, suppose that $0 < t < 1-\theta$. Then, by deleting a finite number of terms of the sequence $\xi(n)$, $n = 1,2,\ldots$, we may assume (using (2)) that there exists a constant $\theta' > \theta$ such that for all n, $\langle q_n\beta\rangle$ is no greater than $1-\theta'$. So, for all n,

$$\mu([\beta, ([q_n\beta] + 1)/q_n)) = ([q_n\beta] + 1)/q_n - \beta$$

$$= (1 - \langle q_n\beta\rangle)/q_n$$

$$\geq \theta'/q_n \tag{6}$$

Now we proceed as follows:

$$\mu(C \cap T_\alpha^{-q_n}C \cap \{x: \sum_{j=0}^{q_n-1} \phi_\beta (T_\alpha^j x) = -\langle q_n\beta\rangle\})$$

$$= \mu(C \cap T_\alpha^{-q_n}C \cap T_\alpha^{-j[q_n\beta])}[\beta,1)) \qquad \text{(by 5)}$$

$$= \mu(T_\alpha^{j[q_n\beta]}(C\cap T_\alpha^{-q_n}C) \cap [\beta,1))$$

$$\geq \mu([\beta, ([q_n\beta] + 1)/q_n))$$

$$-\mu([\,[q_n\beta]/q_n, ([q_n\beta] + 1)/q_n)\setminus T_\alpha^{j([q_n\beta])}(C \cap T_\alpha^{-q_n}C)\,)$$

$$\geq \theta'/q_n - \theta/q_n \qquad \text{(by 4 and 7)}$$

$$= (\theta' - \theta)\,\mu(C). \tag{8}$$

Inequality (8) holds for all n, whenever $C \in \xi(n)$.

Given any $\varepsilon > 0$, we put $\theta_\varepsilon = \theta' - \theta$. By (2) we have $|\langle q_n\beta\rangle - t| < \varepsilon$ for all sufficiently large n. For these n, if $C \in \xi(n)$, then

$$\mu(C \cap T_\alpha^{-q_n} C \cap \{x: |\sum_{j=0}^{q_n-1} \phi_\beta (T_\alpha^j x) - (-t)| < \varepsilon\})$$

$$\geq \mu(C \cap T_\alpha^{-q_n}C \cap \{x: \sum_{j=0}^{q_n-1} \phi_\beta (T_\alpha^j x) = -\langle q_n\beta\rangle\})$$

$$\geq \theta_\varepsilon . \mu(C). \qquad \text{(by 8)}$$

Since the sequence of partitions $\xi(n)$, $n = 1,2,\ldots$, converges to the unit partition as $n \to \infty$, we may now apply Proposition II.3.3 and conclude that $-t \in E_{T_\alpha}(\phi_\beta)$. This is equivalent to

$t \in E_{T_\alpha}(\phi_\beta)$, since $E_{T_\alpha}(\phi_\beta)$ is a subgroup of $\mathbb{R}$ (see II. 3.2).

This completes the proof of assertion (a) of the lemma. We now turn to (b). When $\theta < t < 1$, we can assume without loss of generality that, for some $\theta' > \theta$, we have $\langle q_n\beta \rangle > \theta'$ for all n. This implies that $\mu([\,[q_n\beta]/q_n, \beta)) \geq \theta'/q_n$ for all n. Applying this estimate and using (4) and (5) as in case (a), one obtains

$$\mu(C \cap T_\alpha^{-q_n} C \cap \{x : \sum_{j=0}^{q_n-1} \phi_\beta (T_\alpha^j x) = 1 - \langle q_n\beta \rangle\})$$

$$\geq (\theta' - \theta).\ \mu(C),$$

for all elements C of $\xi(n)$, any n. Another application of Proposition II. 3.3 then yields $1-t \in E_{T_\alpha}(\phi_\beta)$, as required.

The Proof of Proposition 4.1

From Proposition 1.1 we know that there always exists a sequence of fractions p_n/q_n satisfying the conditions of Lemma 4.2 - with $\theta = 1/\sqrt{5}$. Choose an irrational number t such that $1/\sqrt{5} < t < 1 - (1/\sqrt{5})$. Then Lemma 4.2 shows that if β is chosen so that t is a limit point of the sequence $\langle q_n\beta \rangle$, $n = 1,2,\ldots$, then $E_{T_\alpha}(\phi_\beta)$ contains both t and 1-t. In this case, $E_{T_\alpha}(\phi_\beta)$, which is always a closed subgroup of $\mathbb{R}$, must in fact be equal to $\mathbb{R}$; and $T_{\alpha,\beta} = (T_\alpha)_{\phi_\beta}$ must be ergodic (see II. 3.2).

It only remains to apply Proposition 1.3.

The following proposition is taken from Petersen [20].

PROPOSITION 4.3.

For all irrational α, the function ϕ_β is a coboundary with respect to T_α if and only if $\beta = \langle k\alpha \rangle$ for some $k \in Z$.

It follows from this and Proposition II. 1.3 that, for irrational α, the skew products $T_{\alpha,\langle k\alpha\rangle}$, $k \in Z$, are all isomorphic to the trivially non-ergodic $\mathbb{R}$-extension of T_α which maps each $(x,t) \in [0,1) \times \mathbb{R}$ to $(T_\alpha x,t)$. We now consider the more general case when β is rationally dependent on α and 1. Fix irreducible fractions $r_1 = C_1/d_1$ and $r_2 = C_2/d_2$, with d_1 and d_2 positive, and let d be the least common multiple of d_1 and d_2. Suppose that $\beta = \langle r_1\alpha + r_2\rangle$. For this value of β the function $\phi_\beta - r_1\phi_\alpha$ takes its values in the subgroup $d^{-1}.Z$ of $\mathbb{R}$. Now, ϕ_β is cohomologous to $\phi_\beta - r_1\phi_\alpha$ (Proposition 4.3), so that $T_{\alpha,\beta}$ is isomorphic to the skew product $(T_\alpha)_{\phi_\beta - r_1\phi_\alpha}$. From this it follows that $T_{\alpha,\beta}$ is never ergodic when β is rationally dependent on α and 1. Indeed, the space $[0,1) \times \mathbb{R}$ splits up into the disjoint union of the sets $[0,1) \times (d^{-1}Z + t)$, $t \in [0,d^{-1})$, each of which is invariant under $(T_\alpha)_{\phi_\beta - r_1\phi_\alpha}$.

We are therefore led to examine the restriction of $(T_\alpha)_{\phi_\beta - r_1\phi_\alpha}$ to the invariant subspace $[0,1) \times d^{-1}.Z$. Thus, for α irrational, $r_1 = c_1/d_1$ and $r_2 = c_2/d_2$, we define

$$S_{\alpha,\, \langle r_1\alpha+r_2\rangle}(x,t)$$

$$= (x + \alpha(\text{mod } 1),\ t + \phi_{\langle r_1\alpha+r_2\rangle}(x) - r_1\phi_\alpha(x)),$$

$$\text{for all } (x,t) \in [0,1) \times d^{-1}.Z.$$

Note that in the case when $r_1 = 0$, this definition is in accord with the notation used in the previous section.

PROPOSITION 4.4.

Suppose that at least one of the rational numbers $r_1 = c_1/d_1$ and $r_2 = c_2/d_2$ is not an integer. Then, for almost all α in $(0,1)$, the skew product $S_{\alpha,\langle r_1\alpha+r_2\rangle}$ is ergodic.

Proof

As before, we use d to denote the least common multiple of d_1 and d_2. It follows from II. 3.2 that to prove the proposition it is enough to show that $E_{T_\alpha}(\phi_{\langle r_1\alpha+r_2\rangle}) = d^{-1}.Z$, for almost all $\alpha \in (0,1)$.

Recall Proposition 1.2. We now show that for almost all $\alpha \in (0,1)$ there exists a sequence of irreducible fractions, p_n/q_n, $n = 1,2,\ldots$, which, as well as satisfying the first two conditions,

(i) $q_n \nearrow \infty$, as $n \to \infty$,

and

(ii) $q_n^2\, |\alpha - p_n/q_n| \to 0$, as $n \to \infty$,

of 1.2, also has the property:

(iii)' the sequence $\langle q_n(r_1\alpha + r_2)\rangle$, $n = 1,2,\ldots$, has a non-zero limit point $t \in \{1/d, 2/d, \ldots, (d-1)/d\}$.

When r_1 is an integer, one simply requires that $q_n \not\equiv 0 \pmod{d_2}$, for all n. When r_1 is not an integer, it is sufficient to require that $q_n \equiv 0 \pmod{d}$, for all n. In each of these two cases, condition (iii)' follows from (iii) of Proposition 1.2, and the fact that $\langle q_n(r_1\alpha + r_2)\rangle$ is asymptotic to $\langle p_n r_1 + q_n r_2\rangle$ as $n \to \infty$.

We can now apply Lemma 4.2 and conclude that $E_{T_\alpha}(\phi_{\langle r_1\alpha+r_2\rangle})$ contains both t and 1-t. The fact that $E_{T_\alpha}(\phi_{\langle r_1\alpha+r_2\rangle})$ is a group implies that it is equal to $m^{-1}.Z$ for some integer m which divides d. From Proposition II. 3.2(d), we deduce that there exists a measurable function $\psi: [0,1) \to \mathbb{R}$ with

$$\phi_{\langle r_1\alpha+r_2\rangle}(x) - (\psi(T_\alpha x) - \psi(x)) \in m^{-1}.Z,$$

for all $x \in [0,1)$. Hence,

$$y \circ T_\alpha = \exp(-2\pi i m \langle r_1\alpha + r_2\rangle)\, y,$$

where $y = \exp(2\pi i m\, \psi(.))$. So, $\exp(-2\pi i m \langle r_1\alpha + r_2\rangle)$ is an eigenvalue of T_α. This is only possible if each of d_1 and d_2 divides m. Since m divides d, the least common multiple of d_1 and d_2, it follows that d = m. Hence, $E_{T_\alpha}(\phi_{\langle r_1\alpha+r_2\rangle}) = d^{-1}.Z$, as required.

<u>Remark 4.5.</u>

Proposition 4.4 is closely related to Theorem 5 of Conze [3]. In the same paper Conze showed that the transformations $S_{\alpha,\frac{1}{2}}$ are ergodic for all irrational α. This result had previously been obtained by Schmidt [21] for those α with bounded partial quotients in their continued fraction expansion, and by Krygin [15] for those

α for which there exist irreducible fractions, p_n/q_n, with $q_n^3 | \alpha - p_n/q_n| \to 0$.

APPENDIX - A PROOF OF PROPOSITION III. 1.3.

For ease of reference, let us first repeat the statement of the proposition.

PROPOSITION III.1.3.

Let $q_1, q_2, \ldots$ be an arbitrary strictly increasing sequence of positive integers, and let f be a non-negative function defined on the set of positive integers. Suppose that $f(n) \to 0$ as $n \to \infty$.

Then, given any $t \in [0,1]$, those real numbers $\beta \in (0,1)$ for which there exist subsequences, q_{n_k}, $k = 1,2,\ldots$, of the given sequence of positive integers such that

(i) $\langle q_{n_k} \beta \rangle \to t$, as $k \to \infty$,

and

(ii) $|\langle q_{n_k} \beta \rangle - t| > f(q_{n_k})$, for all k,

form a residual subset of full measure in (0,1).

The Proof

For arbitrary $\varepsilon > 0$, let

$$B_\varepsilon(n) = \{\beta \in (0,1): f(q_n) < |\langle q_n \beta \rangle - t| < \varepsilon\},$$

for all $n = 1,2,\ldots$.

Assume for now that t is neither zero nor one. Then, if ε is sufficiently small and n is sufficiently large,

$$B_\varepsilon(n) = \left[\bigcup_{j=0}^{q_n-1} \left(\frac{j+t-\varepsilon}{q_n}, \frac{j+t+\varepsilon}{q_n}\right)\right] \setminus \left[\bigcup_{j=0}^{q_n-1} \left(\frac{j+t-f(q_n)}{q_n}, \frac{j+t+f(q_n)}{q_n}\right)\right].$$

Since $f(q_n) \to 0$ as $n \to \infty$, it follows that $B_\varepsilon(n)$ is asymptotically equal to the union of the q_n uniformly spaced open intervals $((j+t-\varepsilon)/q_n, (j+t+\varepsilon)/q_n)$, $j \in \{0,1,\dots,q_n-1\}$, each of which has length $2\varepsilon/q_n$. Hence, $\mu(B_\varepsilon(n)) \to 2\varepsilon$ as $n \to \infty$. Furthermore, by first showing that

$$\lim_{n\to\infty} \mu(B \cap (\bigcup_{j=0}^{q_n-1} ((j+t-\varepsilon)/q_n, (j+t+\varepsilon)/q_n)))$$

$$= 2\varepsilon\ \mu(B),$$

whenever B is a finite union of intervals, it is a routine matter to show that

$$\lim_{n\to\infty} \mu(B \cap B_\varepsilon(n)) = 2\varepsilon\ \mu(B),$$

for arbitrary measurable $B \subset (0,1)$ (i.e. that $B_\varepsilon(1)$, $B_\varepsilon(2)$, ... is a 2ε-mixing sequence of subsets of $(0,1)$). We may now apply an argument of Fischler [6]:

For J a finite subset of the positive integers, let

$$A_{J,\varepsilon} = (\bigcap_{j\in J} B_\varepsilon(j)) \cap (\bigcap_{j\notin J} B_\varepsilon(j)^c).$$

Then $B_\varepsilon(n) \cap A_{J,\varepsilon} = \emptyset$ for all sufficiently large n. Hence, from the identity

$$\lim_{n\to\infty} \mu(A_{J,\varepsilon} \cap B_\varepsilon(n)) = 2\varepsilon\ \mu(A_{J,\varepsilon})$$

it follows that $\mu(A_{J,\varepsilon}) = 0$, for all J and ε .

Now,

$$(\limsup_{n\to\infty} B_\varepsilon(n))^c = \bigcup_{\substack{J\subset Z^+\\ |J|<\infty}} A_{J,\varepsilon} \quad ,$$

a countable union of sets of measure zero. Hence, for all $\varepsilon > 0$,

$$\mu(\limsup_{n \to \infty} B_\varepsilon(n)) = 1.$$

Observe that, for each $m = 1,2,\ldots$, $\bigcup_{n \geq m} B_\varepsilon(n)$ is open and dense. Hence, the set $\limsup_{n \to \infty} B_\varepsilon(n)$ $\left(= \bigcap_{m=1}^{\infty} \left(\bigcup_{n \geq m} B_\varepsilon(n)\right) \right)$ is also a residual subset of $(0,1)$, for all $\varepsilon > 0$.

To complete the proof, it only remains to note that, for the existence of a subsequence of $q_1, q_2, \ldots$ with the specified properties, it is necessary and sufficient that

$$\beta \in \bigcap_{\ell=2}^{\infty} (\limsup_{n \to \infty} B_{1/\ell}(n)).$$

The argument for the cases $t = 0$ and $t = 1$ is just as above, except that $B_\varepsilon(n)$, $n = 1,2,\ldots$ becomes an ε-mixing sequence of sets.

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